10 Using Binary Systems to Determine Masses and Radii

Let us next consider how we can infer the masses of stars, namely through the study of stellar *binary systems.*

It turns out, in fact, that stellar binary (and even triple and quadruple) systems are quite common, so much so that astronomers sometimes joke that "three out of every two stars is (in) a binary". The joke here works because often two stars in a binary are so close together on the sky that we can't actually resolve one star from another, and so we sometimes mistake the light source as coming from a single star, when in fact it actually comes from two (or even more). But even in such close binaries, we can often still tell there are two stars by carefully studying the observed spectrum, and in this case, we call the system a "spectroscopic binary" (see the next subsection, and figure 10.2).

But for now, let's first focus on the simpler example of "visual binaries", a.k.a. "astrometric" binaries (see figure 10.1), since their detection typically requires precise astrometric measurements of small variations of their positions on the sky over time.

10.1 Visual binaries

In visual binaries, monitoring of the stellar positions over years and even decades reveals that the two stars are actually moving around each other, much as the Earth moves around the Sun. Figure 10.1 illustrates the principles behind visual binaries. The time it takes the stars to go around a full cycle, called the orbital period, can then be measured quite directly. Then if we can convert the apparent angular separation into a physical distance apart – e.g. if we know the distance to the system independently through a measured annual parallax for the stars in the system – then we can use Kepler's 3rd law of orbital motion (as generalized by Newton) to measure the total mass of the two stars.

It's actually quite easy to derive the full formula in the simple case of circular orbits that lie in a plane perpendicular to our line of sight. For stars of mass M_1 and M_2 separated by a physical distance a, Newton's law of gravity gives the attractive force each star exerts on the other,

$$F_g = \frac{GM_1M_2}{a^2} \,. \tag{10.1}$$

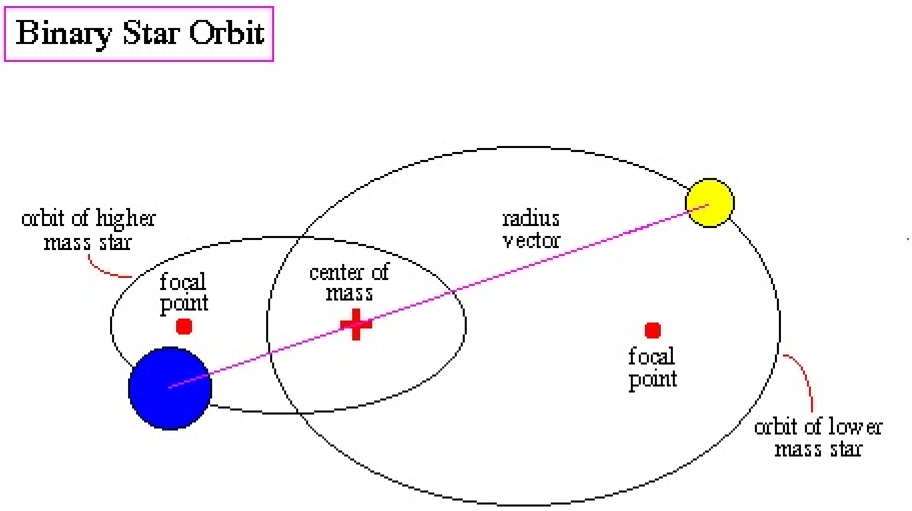

Figure 10.1 Illustration of the properties of a visual binary system.

A key difference from the case of a satellite orbiting the earth, or a planet orbiting a star, is that in binary stars, the masses can become comparable. In this case, each star (1,2) now moves around the *center of mass* at a fixed distance a_1 and a_2, with their ratio given by $a_2/a_1 = M_1/M_2$ and their sum by $a_1 + a_2 = a$. In terms of the full separation, the orbital distance of, say, star 1 is thus given by

$$a_1 = a\frac{M_2}{M_1 + M_2} \,. \tag{10.2}$$

For the given period P, the associated orbital speeds for star 1 is is given by $V_1 = 2\pi a_1/P$. For a stable, circular orbit, the outward centrifugal force on star 1,

$$F_{c1} = \frac{M_1 V_1^2}{a_1} = \frac{4\pi^2 M_1 a_1}{P^2} = \frac{4\pi^2 a}{P^2}\frac{M_1 M_2}{M_1 + M_2} \,, \tag{10.3}$$

must balance the gravitational force from eqn. (10.1), yielding

$$\frac{GM_1M_2}{a^2} = \frac{4\pi^2 a}{P^2}\frac{M_1 M_2}{M_1 + M_2} \,. \tag{10.4}$$

This can be used to obtain the sum of the masses,

$$\boxed{M_1 + M_2 = \frac{4\pi^2}{G}\frac{a^3}{P^2} = \frac{a_{au}^3}{P_{yr}^2} M_\odot \,,} \tag{10.5}$$

where the latter equality shows that evaluating the distance in au and the period in years gives the mass in units of the solar mass. For a visual binary in which we can actually see both stars, we can separately measure the two orbital distances, yielding then the mass ratio $M_2/M_1 = a_1/a_2$. The mass for, e.g., star 1 is thus given by

$$M_1 = \frac{a_{au}^3}{(1 + a_1/a_2)\, P_{yr}^2}\, M_\odot \,. \tag{10.6}$$

The mass for star 2 can likewise be obtained if we just swap subscripts 1 and 2.

Equations (10.5) and (10.6) are actually forms of Kepler's 3rd law for planetary motion around the Sun. Setting $M_1 = M_\odot$ and the planetary mass M_2, we first note that for all planets the mass is much smaller than for the Sun, $M_2/M_1 = a_1/a_2 \ll 1$, implying that the Sun only slightly wobbles (mostly to the counter the pull of the most massive planet, namely Jupiter), with the planets thus pretty much all orbiting around the Sun. If we thus ignore M_2 and plug in $M_1 = M_\odot$ in eqn. (10.5), we recover Kepler's third law in (almost) the form in which he expressed it,

$$P_{yr}^2 = a_{au}^3 \,. \tag{10.7}$$

To be precise, Kepler showed that in general the orbits of the planets are actually *ellipses*, but this same law applies in that case we if we replace the circular orbital distance a with the "semi-major axis" of the ellipse. A circle is just a special case of an ellipse, with the semi-major axis just equal to the radius.

In general, of course, real binary systems often have elliptical orbits, which, moreover, lie in planes that are not always normal to the observer line of sight. These systems can still be fully analyzed using the elliptical orbit form of Newton's generalization of Kepler's 3rd law, as derived, e.g. in Ch. 4 of the Astro 45 notes by Bill Press:

http://www.lanl.gov/DLDSTP/ay45/ay45c4.pdf

Indeed, by watching the rate of movement of the stars along the projected orbit, the inclination effect can even be disentangled from the ellipticity.

10.2 Spectroscopic binaries

As noted, there are many stellar binary systems in which the angular separation between the components is too close to readily resolve visually. However, if the orbital plane is not perpendicular to the line of sight, then the orbital velocities of the stars will give a variable Doppler shift to each star's spectral lines. The effect is greatest when the orbits are relatively *close*, and in a plane *containing* the line of sight, conditions which make such *spectroscopic binaries* complement the wide visual binaries discussed above. Figure 10.2 illustrates the basic features of a spectroscopic binary system.

If the two stars are not too different in luminosity, then observations of the combined stellar spectrum show spectral line signatures of both stellar spectra.

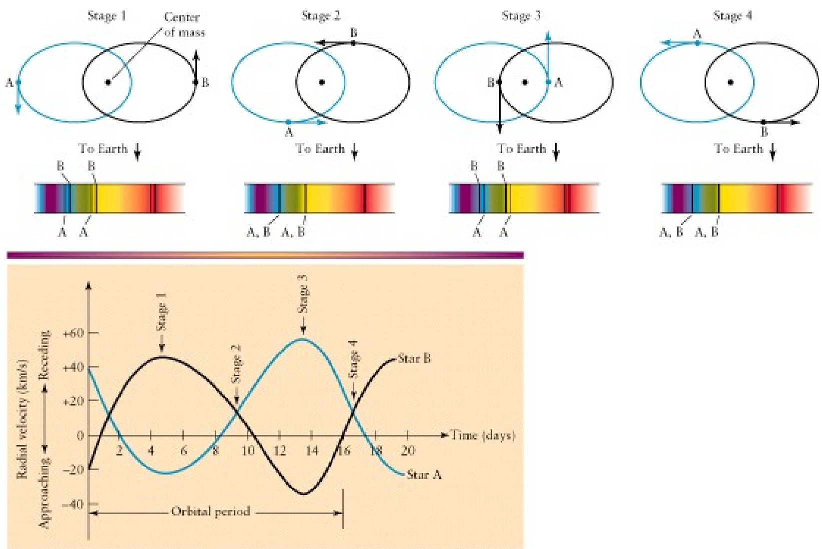

Figure 10.2 Illustration of the periodic Doppler shift of spectral lines in a spectroscopic binary system.

As the stars move around each other in such "double-line" spectroscopic binaries[1], the changing Doppler shift of each of the two spectral line patterns provides information on the changing orbital velocities of the two components, V_1 and V_2.

Considering again the simple case of circular orbits but now in a plane *containing* the line of sight, the inferred radial velocities vary sinusoidally with semi-amplitudes given by the orbital speeds $V_1 = 2\pi a_1/P$ and $V_2 = 2\pi a_2/P$, where a_1 and a_2 are the orbital radii defined earlier. Since the period P is the same for both stars, the ratio of these inferred velocity amplitudes gives the stellar mass ratio, $M_1/M_2 = V_2/V_1$. Using the same analysis as used for visual binaries, but noting now that $a = a_1 + a_2 = PV_1(1 + V_2/V_1)/2\pi$, we obtain a "velocity form" of Kepler's third law given in eqn. (10.6)[2],

$$M_1 = \frac{1}{2\pi G} V_2^3 P \,(1 + V_1/V_2)^2$$
$$M_1 = \left[\frac{V_2}{V_e}\right]^3 P_{yr} \,(1 + V_1/V_2)^2 \, M_\odot \,, \tag{10.8}$$

[1] In "single-line" spectroscopic binaries, the brighter "primary" star is so much more luminous that the lines of its companion are not directly detectable; but this secondary star's presence can nonetheless be inferred from the periodic Doppler shifting of the primary star's lines due to its orbital motion.

[2] In the case that the orbital axis in inclined to the line of sight by an angle i, then these scalings generalize with a factor $\sin^3 i$ multiplying the mass, with the velocities representing the inferred Doppler shifted values.

where the latter equality gives the mass in solar units when the period is evaluated in years, and the orbital velocity in units of the Earth's orbital velocity, $V_e = 2\pi$ au/yr ≈ 30 km/s. Again, an analogous relation holds for the other mass, M_2, if we swap indices 1 and 2.

10.3 Eclipsing binaries

In some (relatively rare) cases of close binaries, the two stars actually pass in front of each other, forming an eclipse that temporarily reduces the amount of light we see. Such eclipsing binaries are often also spectroscopic binaries, and the fact that they eclipse tells us that the inclination of the orbital plane to our line of sight must be quite small, implying that the Doppler shift seen in the spectral lines is indeed a direct measure of the stellar orbital speeds, without the need to correct for any projection effect. Moreover, observation of the eclipse intervals provides information that can be used to infer the individual stellar radii.

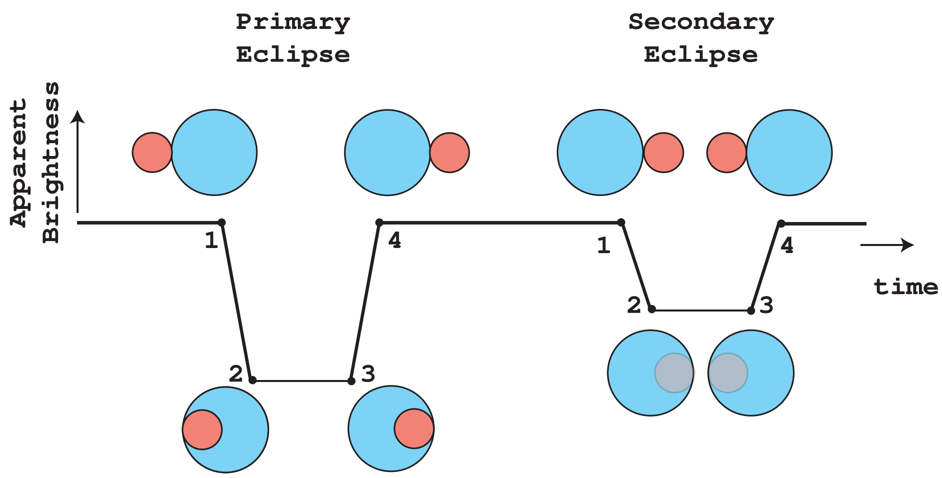

Figure 10.3 Illustration of the how the various contact moments of eclipsing binary star system correspond to features in the observed light curve.

Consider, for example, the simple case that the orbital plane of the two stars is *exactly* in our line of sight, so that the centers of the two pass directly over each other. As noted the maximum Doppler shifts of the lines for each star then gives us a direct measure of their respective orbital speeds, V_1 and V_2. In our above simple example of circular orbits, this speed is constant over the orbit, including during the time when the two stars are moving across our line of sight, as they pass into and out of eclipse. In eclipse jargon, the times when the stellar rims just touch are called "contacts", labeled 1-4 for first, second, etc. Clearly then, once the stellar orbital speeds are known from the Doppler shift, then the radius of the smaller star (R_2) can be determined from the time difference between the first (or last) two contacts,

$$R_2 = (t_2 - t_1)(V_1 + V_2)/2 \,. \tag{10.9}$$

Likewise, the larger radius (R_1) comes from the time between the second and fourth (or third to first) contacts,

$$R_1 = (t_4 - t_2)(V_1 + V_2)/2 \,. \tag{10.10}$$

In principal, one can also use the other, weaker eclipse for similar measurements of the stellar radii.

Of course, in general, the orbits are elliptical and/or tilted somewhat to our line of sight, so that the eclipses don't generally cross the stellar centers, but typically move through an off-center chord, sometimes even just grazing the stellar limb. In these cases information on the radii requires more complete modeling of the eclipse, and fitting the observations with a theoretical light curve that assumes various parameters. Indeed, to get good results, one often has to relax even the assumption that the stars are spheres with uniform brightness, taking into account the mutual tidal distortion of the stars, and how this affects the brightness distributions across their surfaces. Such details are somewhat beyond the scope of this general survey course (but could make the basis for an interesting term paper or project).

10.4 Mass-Luminosity scaling from astrometric and eclipsing binaries

In the above simple introduction of the various types of binaries, we've assumed that the orientation, or "inclination" angle i, of the binary orbit relative to our line of sight is optimal for the type of binary being considered, i.e. looking face on – with $i = 0$ inclination between our sight line and the orbital axis – for the case visual binaries; or edge-on – with $i = 90^o$ – for spectroscopic binaries in which we wish to observe the maximum Doppler shift from the orbital velocities. Of course, in practice binaries are generally at some intermediate, often unknown inclination, leaving an ambiguity in the determination of the mass (typically scaling with $\sin^3 i$) for a given system.

Fortunately, in the relatively few binary systems that are both spectroscopic (with either single or double lines) and either astrometric or eclipsing, it becomes possible to determine the inclination, and so unambiguously infer the *masses* of the stellar components, as well as the *distance* to the system. Together with the observed apparent magnitudes, this thus also gives the associated luminosities of these stellar components.

Figure 10.4 plots $\log L$ vs. $\log M$ (in solar units) for a sample of such astrometric (blue) and eclipsing (red) binaries, showing a clear trend of increasing luminosity with increasing mass. Indeed, a key result is that for many of the stars

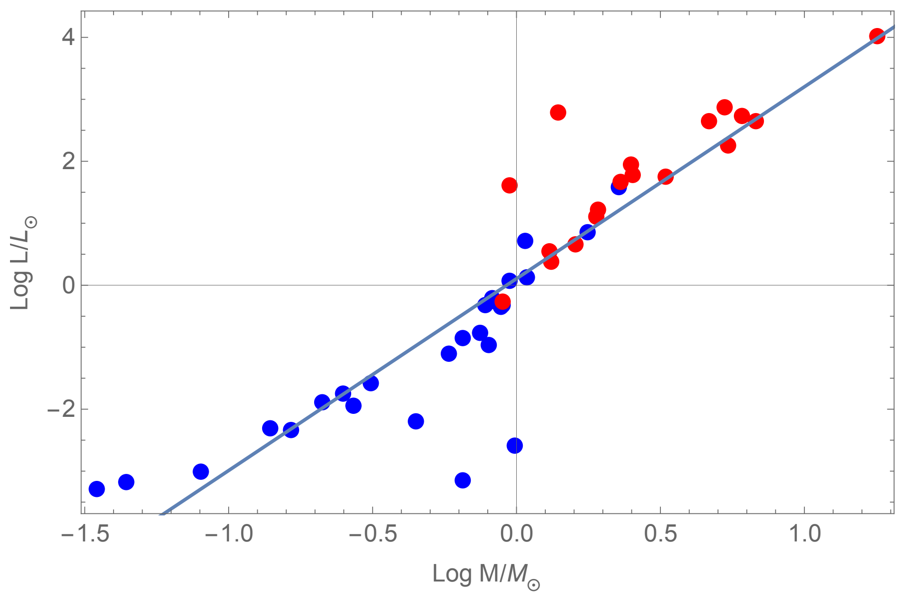

Figure 10.4 A log-log plot of luminosity vs. mass (in solar units) for a sample of 26 astrometric (blue, lower points) binaries and 18 double-line eclipsing (red, upper points) binaries. The best-fit line shown follows the empirical scaling, $\log(L/L_\odot) \approx 0.1 + 3.1 \log(M/M_\odot)$.

(typically those on the main sequence), the data can be well fit by a straight line in this log-log plot, implying a power-law relation between luminosity and mass,

$$\boxed{\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.1}} . \tag{10.11}$$

Part II of these notes will use the stellar structure equations for hydrostatic equilibrium and radiative diffusion to explain why the luminosities of main sequence stars roughly follow this observed scaling with the cube of the stellar mass, $L \sim M^3$ (see §17.1).

10.5 Questions and Exercises

Quick Question 1: Note that the net amount of stellar surface eclipsed is the same whether the smaller or bigger star is in front. So why then is one of the eclipses deeper than the other? What quantity determines which of the eclipses will be deeper?

QQ 2:

Over a period of 10 years, two stars separated by an angle of 1 arcsec are observed