2 Inferring Astronomical Distances

2.1 Angular size

To understand ways we might infer stellar distances, let's first consider how we intuitively estimate distance in our everyday world. Two common ways are through apparent *angular size*, and/or using our *stereoscopic vision*.

For the first, let us suppose we have some independent knowledge of the physical size of a viewed object. The apparent angular size that object subtends in our overall field of view is then used intuitively by our brains to infer the object's distance, based on our extensive experience that a greater distance makes the object subtend a smaller angle.

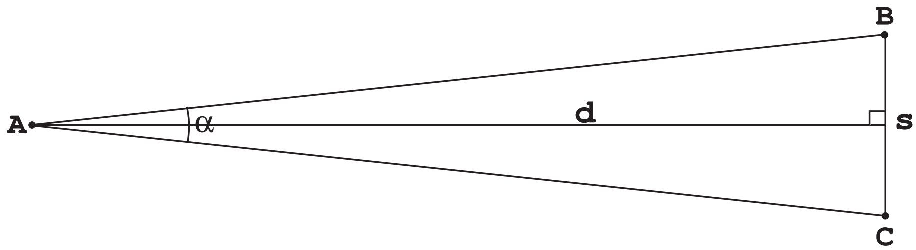

Figure 2.1 Angular size and parallax: The triangle illustrates how an object of physical size s (BC) subtends an angular size α when viewed from a point A that is at a distance d. Note that the same triangle can also illustrate the *parallax* angle α toward the point A at distance d when viewed from two points B and C separated by a length s.

As illustrated in figure 2.1, we can, with the help of some elementary geometry, formalize this intuition to write the specific formula. The triangle illustrates the angle α subtended by an object of size s from a distance d. From simple trigonometry, we find

$$\tan(\alpha/2) = \frac{s/2}{d} \, . \tag{2.1}$$

For distances much larger than the size, $d \gg s$, the angle is small, $\alpha \ll 1$, for which the tangent function can be approximated (e.g. by first-order Taylor expansion) to give $\tan(\alpha/2) \approx \alpha/2$, where α is measured here in radians.

(1 rad = $(180/\pi)^\circ \approx 57^\circ$). The relation between distance, size, and angle thus becomes simply

$$\boxed{\alpha \approx \frac{s}{d}\,.} \tag{2.2}$$

Of course, if we know the physical size and then measure the angular size, we can solve the above relation to determine the distance $d = s/\alpha$.

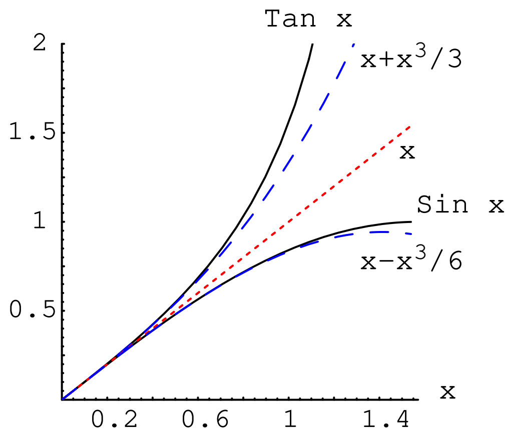

Figure 2.2 Taylor expansion of trig functions $\sin x$ and $\tan x$, about $x = 0$ to order x and order x^3.

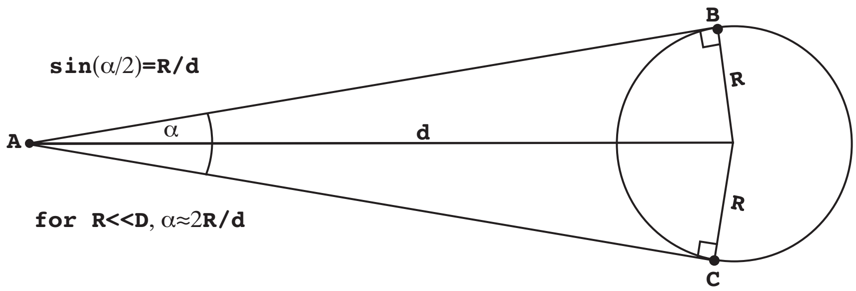

Figure 2.3 Diagram to illustrate the relation between angular size α and diameter $2R$ for a sphere at distance d .

As illustrated in figure 2.3, for a spherical object the angular size α is related to the distance d and radius R through the sine function,

$$\sin(\alpha/2) = R/d\,. \tag{2.3}$$

From figure 2.2 we see that, for the small angles that apply at large distances $d \gg R$, this again reduces to a simple linear form, $\alpha \approx 2R/d$, that relates size to distance.

For example, the distance from the Earth to the Sun, known as an "astronomical unit" (abbreviated "au"), is $d = 1\,\mathrm{au} \approx 150 \times 10^6\,\mathrm{km}$, much larger than the Sun's physical size (i.e. diameter), which is about $s = 2R_\odot \approx 1.4 \times 10^6\,\mathrm{km}$. This means that the Sun has an apparent angular diameter of

$$\alpha_\odot \approx \frac{2R_\odot}{1\mathrm{au}} \approx 0.009 \text{ rad} \approx 0.5^\circ = 30\,\mathrm{arcmin} = 1800\,\mathrm{arcsec}\,. \tag{2.4}$$

However, as noted in §1.2 (and illustrated in figure 1.1), even the nearest stars are more than 200,000 times further away than the Sun. If we assume a similar physical radius (which actually is true for one of the components of the nearest star system, α Centauri A), then

$$\alpha_* = \frac{2R_\odot}{200,000\mathrm{au}} \approx 0.009\,\mathrm{arcsec}\,. \tag{2.5}$$

For ground-based telescopes, the distorting effect of the Earth's atmosphere, known as "atmospheric seeing" (see §13.2), blurs images over an angle size of about 1 arcsec, making it very difficult to infer the actual angular size. There are some specialized techniques, e.g. "speckle interferometry", that can just barely resolve the angular diameter of a few nearby giant stars (e.g. Betelgeuse, a.k.a. α Ori). But generally the difficulty of measuring a star's angular size means that, even if we knew its physical size, we can not use this angular-size method to infer its distance.

2.2 Trignonometric parallax

Fortunately, there is a practical, quite direct way to infer distances to at least relatively nearby stars, namely through the method of *trigonometric parallax.*

This is physically quite analogous to the stereoscopic vision by which we use our two eyes to infer distances to objects in our everyday world. To understand this parallax effect, we can again refer to figure 2.1. If we now identify s as the *separation* between the eyes, then when we view objects at some nearby distance d, the two eyes, in order to combine the separate images as one, have to point inward an angle $\alpha = 2\arctan(s/2d)$. Neurosensors in the eye muscles that effect this inward pointing relay this inward angle to our brain, where it is processed to provide our sense of "depth" (i.e. distance) perception.

You can easily experiment with this effect by placing your finger a few inches from your face, then blinking between your left and right eye, which thus causes the image of your finger to jump back a forth by the angle $\alpha = 2\arctan(s/2d)$. The eye separation s is fixed, but as you move the finger closer and further away, the angle shift will become respectively larger and smaller.

Home Experiment: To illustrate this close link between parallax and angular size, try the following experiment. In front of a wall mirror, close one eye and then extend a finger from either arm to the mirror, covering the image of your closed eye. Without moving your finger, now switch the closure to the other eye. Note that the finger has also switched to cover the other (now closed) eye, even though you didn't physically move it! Note further that this even still works as you decrease the distance from your face to the mirror. The key point here is that the "parallax" angle shift of your finger, which results from switching perspective from one eye to the other, exactly fits the apparent angular separation between your own mirror-image eyes.

Of course, for distances much more than the separation between our eyes, $d \gg s$, the angle becomes too small to perceive, and so we can only use this approach to infer distances of about, say, 10 m. But if we extend the baseline to much larger sizes s, then when coupled with accurate measures of the angle shift α, this method can be used to infer much larger distances.

For example, in the 19th century, there were efforts to use this approach to infer the distance to Mars at time when it was relatively close to Earth, namely at opposition (i.e. when Mars is on the opposite side of the Earth from the Sun). Two expeditions tried to measure the position of Mars at the same time from widely separated sites on Earth. If the distance between the sites is known, the angle difference in the measured directions to Mars, which turns out to be about an arcmin, yields a distance to Mars.

The largest separation possible from two points on the surface of the Earth is limited by the Earth's diameter. But to apply this method of trigonometric parallax to infer distances to stars, we need to use a much bigger baseline than the Earth's diameter. Fortunately though, we don't need then to go into space.

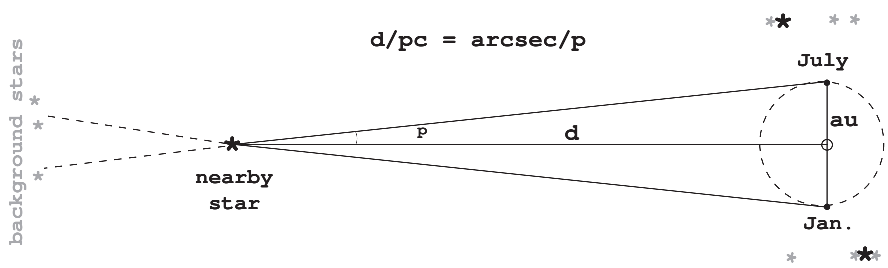

Figure 2.4 Illustration of stellar parallax, in which a relatively nearby star appears to shift against background stars by a parallax angle p as the earth moves through the 1 au radius of earth's orbit. The distance d in parsec (pc) is given by the inverse of p measured in arcsec.

As illustrated in figure 2.4, just waiting a half year from one place on the Earth allows us, as a result of the Earth's *orbit* around the Sun, to view the stars from two points separated by twice the Earth's orbital radius, i.e. 2 au. By convention, however, the associated "parallax angle" α of a star is traditionally quoted in terms of the shift from a baseline s of just *one* au. If we scale the parallax angle in units of an arcsec, the distance is

$$\boxed{d = \frac{s}{\alpha} = \frac{206,265\,\text{arcsec/radian}}{\alpha/\text{radian}}\ \text{au} \equiv \frac{\text{arcsec}}{\alpha}\ \text{parsec}}\,, \tag{2.6}$$

where we note that the conversion between arcsec and radian is given by $(180/\pi)$ degree/radian $\times$ 60 arcmin/degree $\times$ 60 arcsec/arcmin $=$ 206,265 arcsec/radian. In the last equality, we have also introduced the distance unit *parsec* (short for "parallax second", and often further abbreviated as "pc"), which is defined as the distance at which the parallax angle is 1 arcsec. It is thus apparent that $1\,\text{pc} = 206,265\,\text{au}$, which works out to give $1\,\text{pc} \approx 3 \times 10^{16}$ m.

The "parsec" is one of the two most common units used to characterize the huge distances we encounter in astronomy. The other is the *light-year*, which is the distance light travels in a year, at the speed of light $c = 3 \times 10^8\,\text{m/s}$. The number of seconds in a year is given by $1\,\text{yr} = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^7\,\text{s}$, which, coincidentally, can be remembered as $1\,\text{yr} \approx \pi \times 10^7\,\text{s}$ (or since $\sqrt{10} \approx 3.16$, $1\,\text{yr} \approx 10^{7.5}\,\text{s}$). Thus a light-year is roughly $1\,\text{ly} \approx 3\pi \times 10^{8+7} \approx 9.5 \times 10^{15} \approx 10^{16}\,\text{m}$. In terms of parsecs, we can see that $1\,\text{pc} \approx 3.26\,\text{ly}$.

The parallax for even the nearest star is less than an arcsec, implying stars are all at distances more (generally *much* more) than a parsec. By repeated observation, the roughly 1 arcsec overall blurring of single stellar images by atmospheric seeing can be averaged to give a position accuracy of about $\Delta\alpha \approx 0.01$ arcsec, implying that one can estimate distances to stars out to about $d \approx 100$ pc. The Hipparchus satellite orbiting above Earth's atmosphere can measure parallax angles approaching a milliarcsec (1 mas $= 10^{-3}$ arcsec), thus potentially extending distance measurements for stars out to about a kiloparsec, $d \approx 1$ kpc. However, parallax measurements out to such distances typically require a relatively bright source. In practice, only a fraction of all the stars (those with the highest intrinsic brightness, or "luminosity") with distances near $d \approx 1$ kpc have thus far had accurate measurements of their parallax[1].

Again, from the above discussion it should be apparent that parallax is really the "flip slide" of the angular size vs. distance relation. That is, the triangle in figure 2.1 was initially used to illustrate how, from the perspective of a given point A, the angle α subtended by an object is set by the ratio of its size s to its distance d. But if we consider a simple change of observer's perspective to the two *endpoints* (B and C) of the size seqment s, then the same triangle can be used equally well to illustrate the observed parallax angle α for the point A at a distance d.

For the large ($>$ parsec) distances in astronomy, it is convenient to rewrite our simple equation (2.2) to scale angular size in arcsec, with the size in au and distance in pc:

$$\boxed{\frac{\alpha}{\text{arcsec}} = \frac{s/\text{au}}{d/\text{pc}}\,.} \tag{2.7}$$

[1] Since 2013 a follow-up satellite mission call Gaia has been in the process of measuring the absolute position and parallax to roughly one *billion* stars; see `http://sci.esa.int/gaia/`.

2.3 Determining the Astronomical Unit (au)

We thus see that determining the distance of the Earth to the Sun, i.e. measuring the physical length of an au, provides a fundamental basis for determining the distances to stars and other objects in the universe. In modern times, one way this is computed involves first measuring the distance from the Earth to the planet Venus through "radar ranging", i.e. measuring the time Δt it takes a radar signal to bounce off Venus and return to Earth. The associated Earth-Venus distance is then given by

$$d_{EV} = \frac{c\Delta t}{2} \,. \tag{2.8}$$

If this distance is measured at the time when Venus has its "maximum elongation", or maximum angular separation, from the Sun, which is found to be about 47^o, then one can use simple trigonometry to derive a physical value of the au. The details are left as an exercise for the reader. (See Exercise 2-1 at the end of this section.)

2.4 Solid angle

In general objects that have a measurable angular size on the sky are extended in *two* independent directions. As the 2D generalization of an angle along just one direction, it is useful then to define for such objects a 2D *solid angle* Ω, measured now in *square radians*, but more commonly referred by the shorthand "*steradians*".

Just as projected area A is related to the square of physical size s (or radius R), so is solid angle Ω related to the square of the *angular size* α. For an object at a distance d with projected area A, the solid angle is just

$$\Omega = \frac{A}{d^2} \approx \frac{\pi R^2}{d^2} = \pi\alpha^2 \,, \tag{2.9}$$

where the latter equalities assume a sphere (or disk) with projected radius R and associated angular radius $\alpha = R/d$.

For more general shapes, figure 2.5 illustrates how a small solid-angle patch $\delta\Omega$ is defined in terms of ranges in the standard spherical angles representing co-latitude θ and azimuth ϕ on a sphere. An extended object would then have a solid angle given by the integral

$$\Omega = \int d\phi \, \sin\theta \, d\theta \,. \tag{2.10}$$

Integration over a full sphere shows that there are 4π steradians in the full sky. This represents the 2D analog to the 2π radians around the full circumference of a circle.

For our example of a circular patch of angular radius α, let us assume the

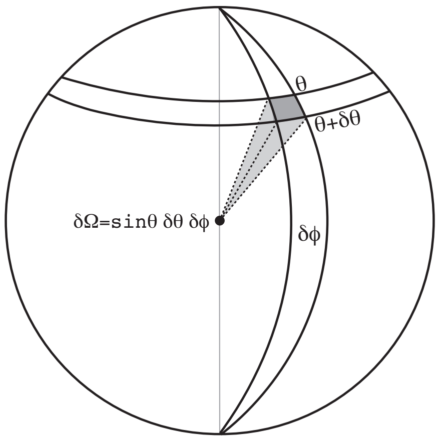

Figure 2.5 Diagram to illustrate a small patch of solid angle $\delta\Omega$ seen by an observer at the center of a sphere, with size defined by ranges in the co-latitude θ and azimuth ϕ.

object is centered around the coordinate pole – representing perhaps the image of a distant spherical object like the Sun or moon. The azimuthal symmetry means the ϕ integral evaluates to 2π, while carrying out the remaining integral over co-latitude range 0 to α then gives

$$\Omega = 2\pi \left[1 - \cos\alpha\right] . \tag{2.11}$$

In particular, applying the angular radius of the Sun $\alpha_\odot \approx R_\odot/\mathrm{au}$ and expanding the cosine to first order (i.e., $\cos x \approx 1 - x^2/2$), we find

$$\Omega_\odot = 2\pi \left[1 - \cos(R_\odot/\mathrm{au})\right] \approx \pi (R_\odot/\mathrm{au})^2 \approx \pi\alpha_\odot^2 . \tag{2.12}$$

One can alternatively measure solid angle in terms of square degrees. Since there are $180/\pi \approx 57.3$ degrees in a radian, there are $(180/\pi)^2 = 57.3^2 \approx 3283$ square degrees in a steradian; the number of square degrees in the 4π steradians of the full sky is thus

$$4\pi \left(\frac{180}{\pi}\right)^2 = 41,253\,\mathrm{deg}^2 . \tag{2.13}$$

The Sun and moon both have angular radii of about 0.25^o, meaning they each have a solid angle of about $\pi(0.25)^2 = \pi/16 = 0.2\,\mathrm{deg}^2 = 6 \times 10^{-5}$ ster, which is about $1/200,000$ of the full sky[2].

2.5 Questions and Exercises

Quick Question 1: A helium party balloon of diameter 20 cm floats 1 meter above your head.

[2] If you think about it, you'll see that this helps explain why a full moon is about a million times dimmer than full sunlight! See Exercise 2-3.