7 Surface Gravity and Escape/Orbital Speed

So far we've been able to finds ways to estimate the first five stellar parameters on our list – distance, luminosity, temperature, radius, and elemental composition. Moreover, we've done this with just a few, relatively simple measurements – parallax, apparent magnitude, color, and spectral line patterns. But along the way we've had to learn to exploit some key geometric principles and physical laws – angular-size/parallax, inverse-square law, and Planck's, Wien's and the Stefan-Boltzman laws of blackbody radiation.

So what of the next item on the list, namely stellar mass? Mass is clearly a physically important parameter for a star, since for example it will help determine the strength of the gravity that tries to pull the star's matter together. To lay the groundwork for discussing one basic way we can determine mass (from orbits of stars in stellar binaries), let's first review the Newton's law of gravitation and show how this sets such key quantities like the surface gravity, and the speeds required for material to escape or orbit the star.

7.1 Newton's law of gravitation and stellar surface gravity

On Earth, an object of mass m has a weight given by

$$F_{grav} = mg_e \,, \tag{7.1}$$

where the acceleration of gravity on Earth is $g_e = 980\, cm/s^2 = 9.8\, m/s^2$. But this comes from Newtons's law of gravity, which states that for two point masses m and M separated by a distance r, the attractive gravitational force between them is given by

$$\boxed{F_{grav} = \frac{GMm}{r^2}} \,, \tag{7.2}$$

where Newton's constant of gravity is $G = 6.7 \times 10^{-8} cm^3/g/s^2$. Remarkably, when applied to spherical bodies of mass M and finite radius R, the same formula works for all distances $r \geq R$ at or outside the surface![1] Thus, we see that the

[1] Even more remarkably, even if we are *inside* the radius, $r < R$, then we can still use Newton's law if we just count that part of the total mass that is *inside* r, i.e. M_r, and completely ignore all the mass that is above r.

acceleration of gravity at the surface of the Earth is just given by the mass and radius of the Earth through

$$g_e = \frac{GM_e}{R_e^2} \,. \tag{7.3}$$

Similarly for stars, the surface gravity is given by the stellar mass M and radius R. In the case of the Sun, this gives $g_\odot = 2.6 \times 10^4\,\mathrm{cm/s^2} \approx 27\,g_e$. Thus, if you could stand on the surface of the Sun, your "weight" would be about 27 times what it is on Earth.

For other stars, gravities can vary over a quite wide range, largely because of the wide range in size. For example, when the Sun get's near the end of its life about 5 billion years from now, it will swell up to more than 100 times its current radius, becoming what's known as a "Red Giant" (§19). Stars we see now that happen to be in this Red Giant phase thus tend to have quite low gravity, about a fraction 1/10,000 that of the Sun.

Largely because of this very low gravity, much of the outer envelope of such Red Giant stars will actually be lost to space (forming, as we shall see, quite beautiful nebulae; see §19 and figure 20.5.) When this happens to the Sun, what's left behind will be just the hot stellar core, a so-called "white dwarf", with about 2/3 the mass of the current Sun, but with a radius only about that of the Earth, i.e. $R \approx R_e \approx 7 \times 10^3\, km \approx 0.01\, R_\odot$. The surface gravities of white dwarfs are thus typically $10,000$ times *higher* than the current Sun (§19.4).

For "neutron stars", which are the remnants of stars a bit more massive than the Sun, the radius is just about 10 km, more than another factor 500 smaller than white dwarfs (§20.3). This implies surface gravities another 5-6 orders of magnitude higher than even white dwarfs. (Imagine what you'd weigh then on the surface of a neutron star!)

Since stellar gravities vary over such a large range, it is customary to quote them in terms of the log of the gravity, $\log g$, using CGS units. We thus have gravities ranging from $\log g \approx 0$ for Red Giants, to $\log g \approx 4$ for normal stars like the Sun, to $\log g \approx 8$ for white dwarfs, to $\log g \approx 13$ for neutron stars. Since the Earth's gravity has $\log g_e \approx 3$, the difference of $\log g$ from 3 is the number of order of magnitudes more/less that you'd weigh on that surface. For example, for neutron stars the difference from Earth is 10, implying you'd weigh 10^{10}, or ten billion times more on a neutron star! On the other hand, on a Red Giant, your weight would be about 1000 times *less* than on Earth.

7.2 Surface escape speed V_{esc}

Another measure of the strength of a gravitational field is through the surface escape speed,

$$\boxed{V_{esc} = \sqrt{\frac{2GM}{R}}\,.} \tag{7.4}$$

A object of mass m launched with this speed has a kinetic energy $mV_{esc}^2/2 = GMm/R$. This just equals the work needed to lift that object from the surface radius R to escape at a large radius $r \to \infty$,

$$W = \int_R^\infty \frac{GMm}{r^2}\, dr = \frac{GMm}{R}\,. \tag{7.5}$$

Thus if one could throw a ball (or launch a rocket!) with this speed outward from a body's surface radius R, then[2] by conservation of total energy, that object would reach an arbitrarily large distance from the star, with however a vanishingly small final speed.

For the earth, the escape speed is about 25,000 mph, or 11.2 km/s. By comparison, for the moon, it is just 2.4 km/s, which is one reason the Apollo astronauts could use a much smaller rocket to get back from the moon, than they used to get there in the first place. However, escaping from the surface of the Sun (and most any star), is *much* harder, requiring an escape speed of 618 km/s.

7.3 Speed for circular orbit

Let us next compare this escape speed with the speed needed for an object to maintain a circular orbit at some radius r from the center a gravitating body of mass M. For an orbiting body of mass m, we require that the gravitational force be balanced by the centrifugal force from moving along the circle of radius r,

$$\frac{GMm}{r^2} = \frac{mV_{orb}^2}{r}\,, \tag{7.6}$$

which solves to

$$\boxed{V_{orb}(r) = \sqrt{\frac{GM}{r}}\,.} \tag{7.7}$$

Note in particular that the orbital speed very near the stellar surface, $r \approx R$, is given by $V_{orb}(R) = V_{esc}/\sqrt{2}$. Thus the speed of satellites in low-earth-orbit (LEO) is about 17,700 mph, or 7.9 km/s.

Of course, orbits can also be maintained at any radius above the surface radius, $r > R$, and eqn. (7.7) shows that in this case, the speed needed declines as $1/\sqrt{r}$. Thus, for example, the orbital speed of the earth around the Sun is about 30 km/s, a factor of $\sqrt{R_\odot/au} = \sqrt{1/215} = 0.0046$ smaller than the orbital speed near the Sun's surface, $V_{orb,\odot} = 434$ km/s.

7.4 Virial Theorum for bound orbits

If we define the gravitational energy to be zero far from a star, then for an object of mass m at a radius r from a star of mass M, we can write the gravitational

[2] neglecting forces other than gravity, like the drag from an atmosphere

binding energy U as the *negative* of the escape energy,

$$U(r) = -\frac{GMm}{r} \, . \tag{7.8}$$

If this same object is in orbit at this radius r, then the kinetic energy of the orbit is

$$T(r) = \frac{mV_{orb}^2}{2} = +\frac{GMm}{2r} = -\frac{U(r)}{2} \, , \tag{7.9}$$

where the second equation uses eqn. (7.7) for the orbital speed $V_{orb}(r)$. We can then write the *total* energy as

$$E(r) \equiv T(r) + U(r) = -T(r) = \frac{U(r)}{2} \, . \tag{7.10}$$

This fact that the total energy E just equals *half* the gravitational binding energy U is an example of what is known as the *Virial Theorum.* It is applicable broadly to most any stably bound gravitational system. For example, if we recognize that the thermal energy inside a star as a kind of kinetic energy, it even applies to stars, in which the internal gas pressure balances the star's own self gravity. This is discussed further in §8.2 and the part II notes on stellar structure.

7.5 Questions and Exercises

Quick Question 1: In CGS units, the Sun has $\log g_\odot \approx 4.44$. Compute the $\log g$ for stars with:
a. $M = 10 M_\odot$ and $R = 10 R_\odot$
b. $M = 1 M_\odot$ and $R = 100 R_\odot$
c. $M = 1 M_\odot$ and $R = 0.01 R_\odot$

Quick Question 2:
The Sun has an escape speed of $V_{e\odot} = 618\,\text{km/s}$. Compute the escape speed V_e of the stars in parts a-c of QQ1.

Quick Question 3:
The earth has an orbital speed of $V_e = 2\pi\text{au/yr} = 30\,\text{km/s}$. Compute the orbital speed V_{orb} (in km/s) of a body at the following distances from the stars with the quoted masses:
a. $M = 10 M_\odot$ and $d = 10\,\text{au}$.
b. $M = 1 M_\odot$ and $d = 100\,\text{au}$.
c. $M = 1 M_\odot$ and $d = 0.01\,\text{au}$.

Exercise 1:
a. During a solar eclipse, the moon just barely covers the visible disk of the Sun. What does this tell you about the relative angular size of the Sun and moon?
b. Given that the moon is at a distance of 0.0024 au, what then is the ratio of the *physical* size of the moon vs. Sun?