6. Evolution and internal structure of the Sun. I.

Formation and Evolution of the Sun

1. Determination of the Sun's age
2. Pre-main sequence evolution
3. Jeans instability
4. Kelvin-Helmholtz stage
5. Evolution of the Sun on the HR diagram
6. Equations of stellar structure
7. Hydrostatic equations
8. Radiative energy transfer
9. Equation of state
10. Nuclear reactions

Sun's age

The age of the Sun is estimated from meteorites, the oldest bodies in the solar system. Their age is determined from the decay of radioactive isotopes, such as ^{87}Rb (Rubidium) which has a half-life of 4.8×10^{10} years. It decays into stable isotope Strontium (^{87}Sr).

$$^{87}\mathrm{Sr}_{\mathrm{now}} = {}^{87}\mathrm{Sr}_{\mathrm{original}} + ({}^{87}\mathrm{Rb}_{\mathrm{original}} - {}^{87}\mathrm{Rb}_{\mathrm{now}})$$

$$^{87}\mathrm{Rb}_{\mathrm{original}} = {}^{87}\mathrm{Rb}_{\mathrm{now}} \times e^{\lambda t},$$

Radioactive decay

$$^{87}\mathrm{Sr}_{\mathrm{now}} = {}^{87}\mathrm{Sr}_{\mathrm{original}} + {}^{87}\mathrm{Rb}_{\mathrm{now}} \times \left(e^{\lambda t} - 1\right).$$

$$\frac{^{87}\mathrm{Sr}_{\mathrm{now}}}{^{86}\mathrm{Sr}} = \frac{^{87}\mathrm{Sr}_{\mathrm{original}}}{^{86}\mathrm{Sr}} + \frac{^{87}\mathrm{Rb}_{\mathrm{now}}}{^{86}\mathrm{Sr}} \times \left(e^{\lambda t} - 1\right).$$

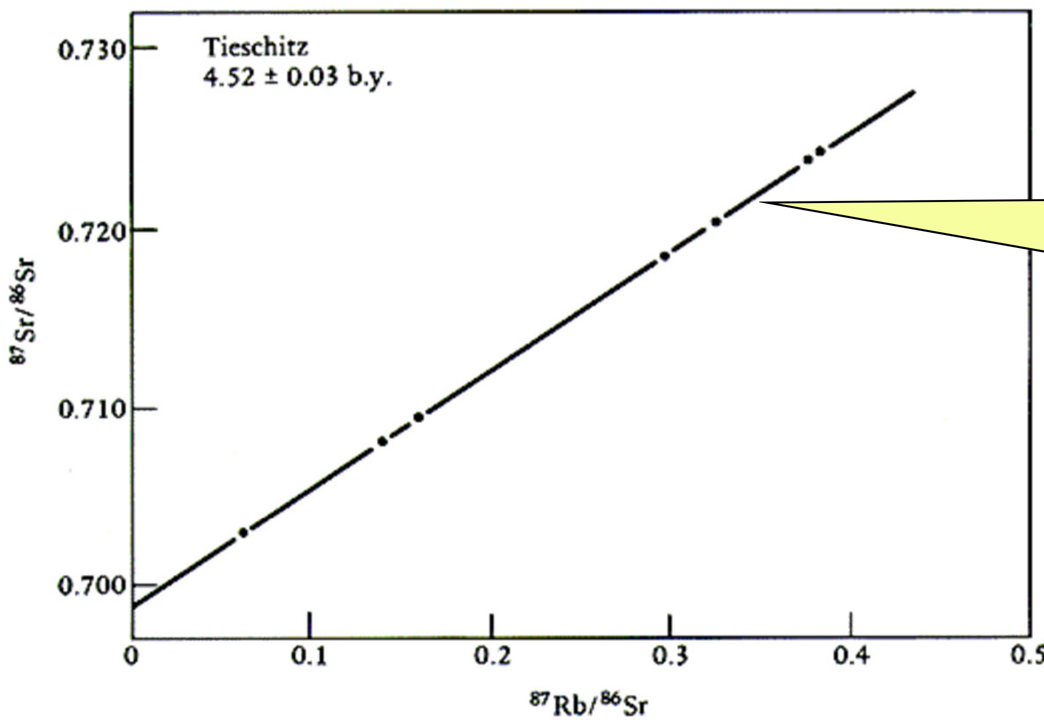
Does not change over time

Measured by mass spectrometer

$$\frac{^{87}\mathrm{Sr}_{\mathrm{now}}}{^{86}\mathrm{Sr}} = \frac{^{87}\mathrm{Sr}_{\mathrm{original}}}{^{86}\mathrm{Sr}} + \frac{^{87}\mathrm{Rb}_{\mathrm{now}}}{^{86}\mathrm{Sr}} \times \left(e^{\lambda t} - 1\right).$$

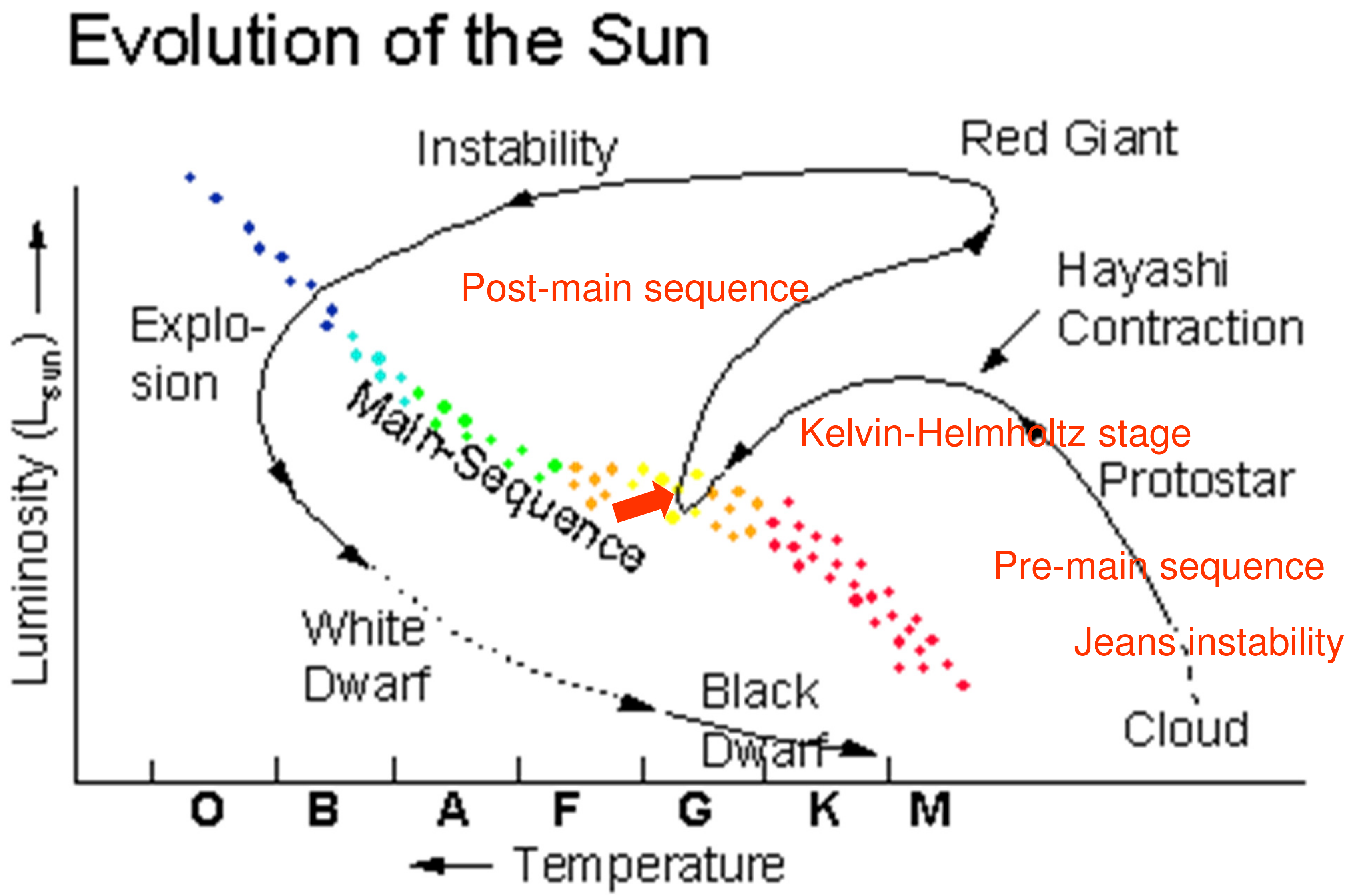
Note that this is the equation of a line in the form

$$y = b + x \cdot m \qquad m = \left(e^{\lambda t} - 1\right)$$



The age is determined from the slope

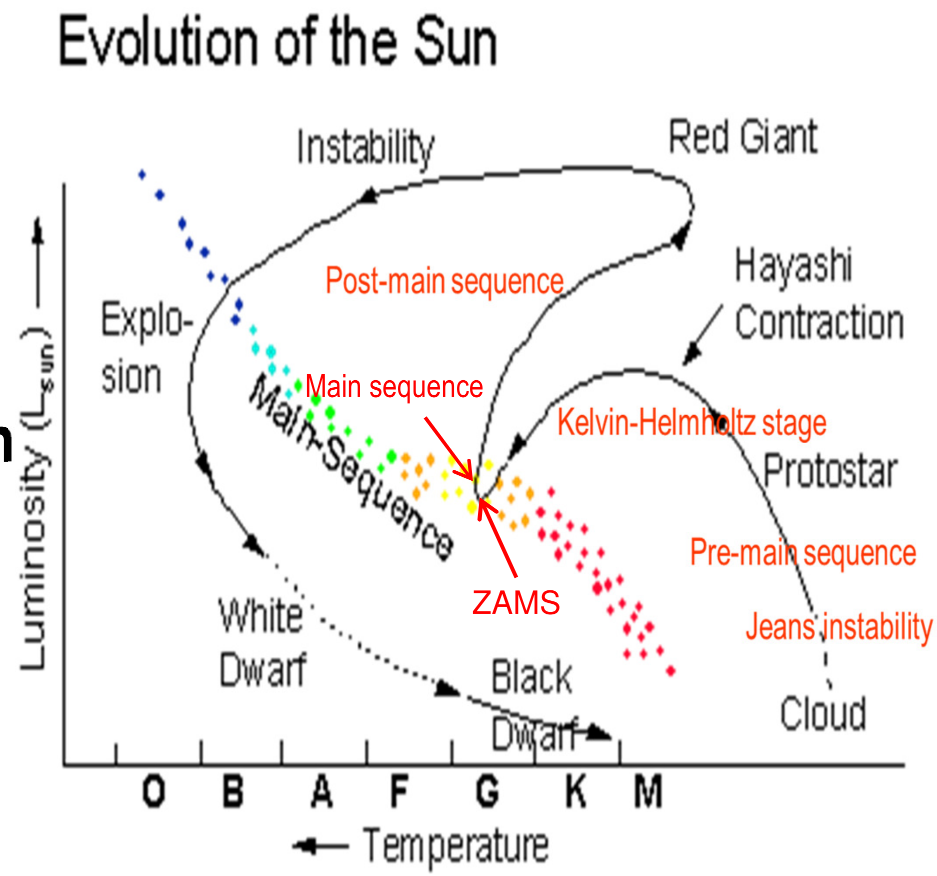
Precise measurement of radioactive isotopes provides a way to determine the age of a meteorite. In the method illustrated here, a radioactive isotope of rubidium (^{87}Rb) decays to a stable isotope of strontium (^{87}Sr), which mixes with already existing strontium isotopes (^{87}Sr and ^{86}Sr). The individual data points in this diagram represent minerals separated from the Tieschitz (Czechoslovakia) chondrite. The age of the meteorite (4.52 billion years) is calculated from the slope of the diagonal line, which steepens with increasing time.


Evolution of the Sun
Instability
Red Giant
Hayashi
Contraction
Post-main sequence
Explo-
sion
Main-Sequence
Kelvin-Helmholtz stage
Protostar
Pre-main sequence
Jeans instability
White
Dwarf
Black
Dwarf
Cloud
Luminosity (L_{sun})
O B A F G K M
Temperature

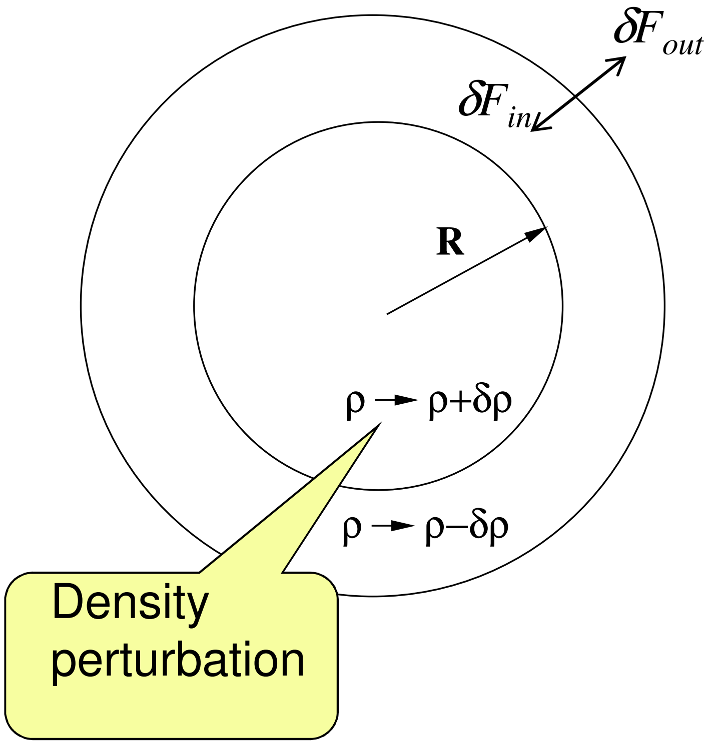
HR diagram

Stages of the solar evolution

- **Pre-main sequence evolution**
  - Jeans instability
  - Kelvin-Helmholtz stage
  - Zero-age main sequence (ZAMS)
- **Main sequence evolution**
- **Post-main sequence evolution**
  - Red giant
  - Helium flash
  - White dwarf
  - Black dwarf



Jeans instability



Excess of gravity force:

$$\delta F_{\text{in}} = \frac{G\delta M}{R^2}\rho = \frac{G\rho}{R^2}\left(\frac{4}{3}\pi R^3 \delta\rho\right),$$

Excess of pressure force:

$$\delta F_{\text{out}} = \frac{\delta P}{R} \sim c_T^2 \frac{\delta\rho}{R},$$

$$\delta P \sim \frac{P}{\rho}\delta\rho \sim c_T^2 \delta\rho$$

$$T = const$$

$$P = \Re \rho T$$

$$c_T^2 = \Re T$$

↑ Gas constant

The instability (collapse) occurs when for $\delta\rho > 0$:

$$\delta F_{\text{in}} > \delta F_{\text{out}}.$$

Jeans instability

The instability (collapse) occurs when for $\delta\rho > 0$:

$\delta F_{\text{in}} > \delta F_{\text{out}}$.

$$\delta F_{\text{in}} = \frac{G\rho}{R^2}\left(\frac{4}{3}\pi R^3 \delta\rho\right), \qquad \delta F_{\text{out}} = c_T^2 \frac{\delta\rho}{R},$$

$$\frac{G\rho}{R^2}\left(\frac{4}{3}\pi R^3 \delta\rho\right) > c_T^2 \frac{\delta\rho}{R} \quad \Longrightarrow \quad \frac{4\pi}{3} G\rho R^2 > c_T^2$$

$$R > \frac{c_T}{\sqrt{\frac{4\pi}{3} G\rho}} \equiv R_J,$$

Jeans radius

Example

Consider an interstellar cloud of temperature $T = 50$ K,
density $\rho = 10^{-20}$ g cm^{-3}.
Then,

$$c_T = \sqrt{\Re T} \approx \sqrt{8.3\times 10^7 \cdot 50} \simeq 2\times 10^4 \text{ cm/s}$$

($\Re$=8.314×10^7 erg K^{-1} mol^{-1} is the gas constant),

$$R > R_J \equiv \frac{c_T}{\sqrt{\frac{4\pi}{3} G\rho}} = \frac{2\times 10^4}{\sqrt{4\cdot 6.67\times 10^{-8}\cdot 10^{-20}}} \simeq 4\times 10^{17} \text{ cm.}$$

This is approximately equal to 0.4 light years.

The corresponding critical (Jeans) mass is:

$$M_J = \frac{4\pi}{3}\rho R_J^3 = \frac{4\pi}{3}\rho \frac{c^3}{\left(\frac{4\pi}{3}G\rho\right)^{3/2}} = \left(\frac{4\pi}{3}\rho\right)^{-1/2} \frac{c^3}{G^{3/2}} \sim 2\times 10^{33}\text{g}$$

The characteristic time of collapse (free-fall) can be estimated from:

$$R \sim \frac{gt^2}{2} = \frac{GM}{2R^2} t^2$$

.

$$t \sim \sqrt{2\frac{R^3}{GM}} \sim \frac{1}{\sqrt{G\rho}} \sim 1.2\times10^6 \text{ years.}$$

Rotation and magnetic field play important role in the collapse.
The magnetic field braking removes the angular momentum.

During the collapse the Jeans radius decreases as

$$R_J \propto \rho^{-1/2} \propto R^{3/2}, \qquad (\rho R^3 = const)$$

where R is a characteristic radius of the collapsing cloud. Since the critical radius decreases faster than R the cloud fragments into smaller parts.

The fragmentation stops when the cloud becomes optically thick, and the process changes from isothermal to adiabatic. In this case, the sound speed:

$$c \propto (P/\rho)^{1/2} \propto \rho^{(\gamma-1)/2} \quad \text{(because } P \propto \rho^\gamma\text{)},$$

and then

$$R_J \propto \rho^{(\gamma-1)/2-1/2} \propto R^\alpha,$$

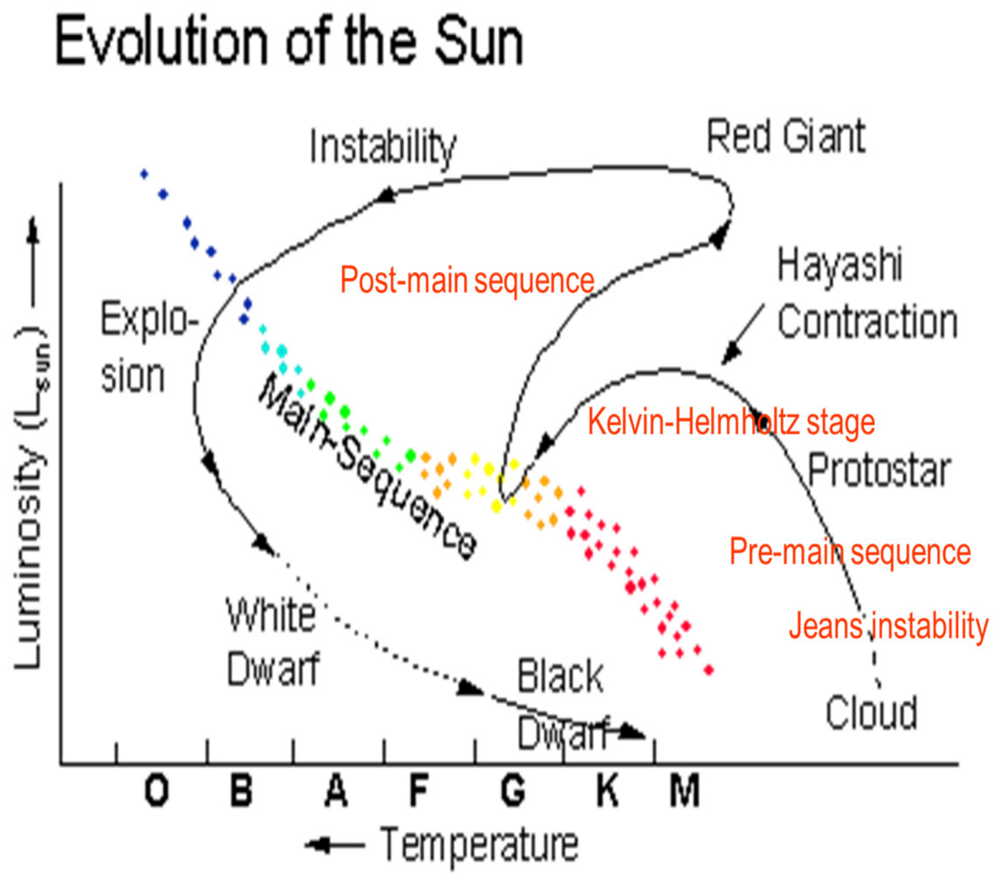
where $\alpha = \frac{3}{2}(2-\gamma)$.

The fragmentation stops when $\alpha < 1$, that is when $\gamma > 4/3$.

Kelvin-Helmholtz stage

When the temperature rises the contraction slows because of the increase of the internal pressure. The protostar reaches hydrostatic equilibrium.

A slow contraction (Kelvin-Helmholtz) stage begins. During this stage the gravitational energy is released in the form of the radiated energy and the internal energy of the star.



Kelvin-Helmholtz stage

According to the virial theorem the release of the gravitational energy is equally divided between the thermal energy and the energy radiated into space. As the star tries to cool it gets hotter!

The characteristic time of the K-H stage is:

$$t_{\mathrm{KH}} = \frac{GM^2}{RL_\odot} \sim \frac{6.67\times10^{-8}\cdot(2\times10^{33})^2}{7\times10^{10}\cdot 4\times10^{33}} \sim 10^{15}\ \mathrm{s} \sim 3\times10^{7}\ \text{years.}$$

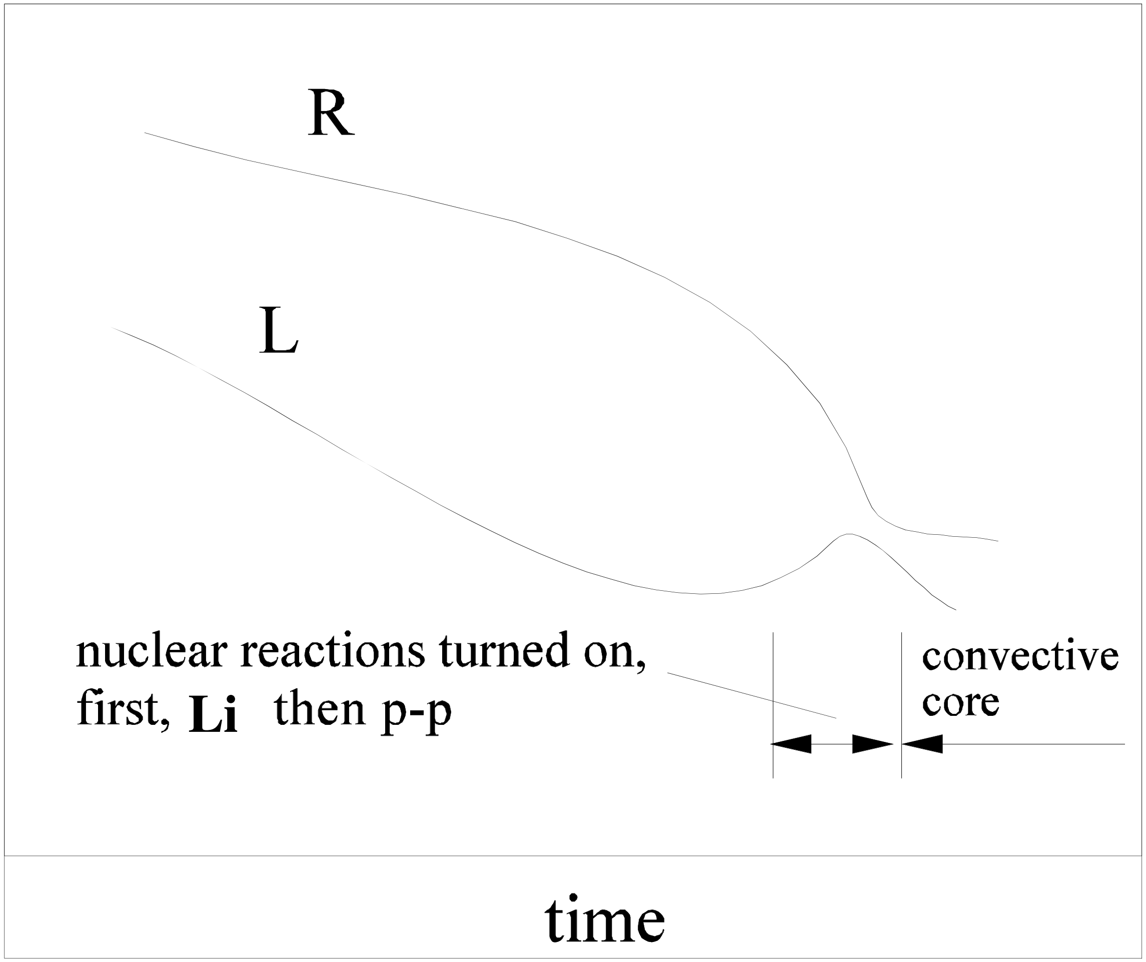
$$G = 6.67\times10^{-8}\, cm^3\cdot g^{-1}\cdot s^{-2}$$

$$M = 1.988\times10^{33}\, g$$

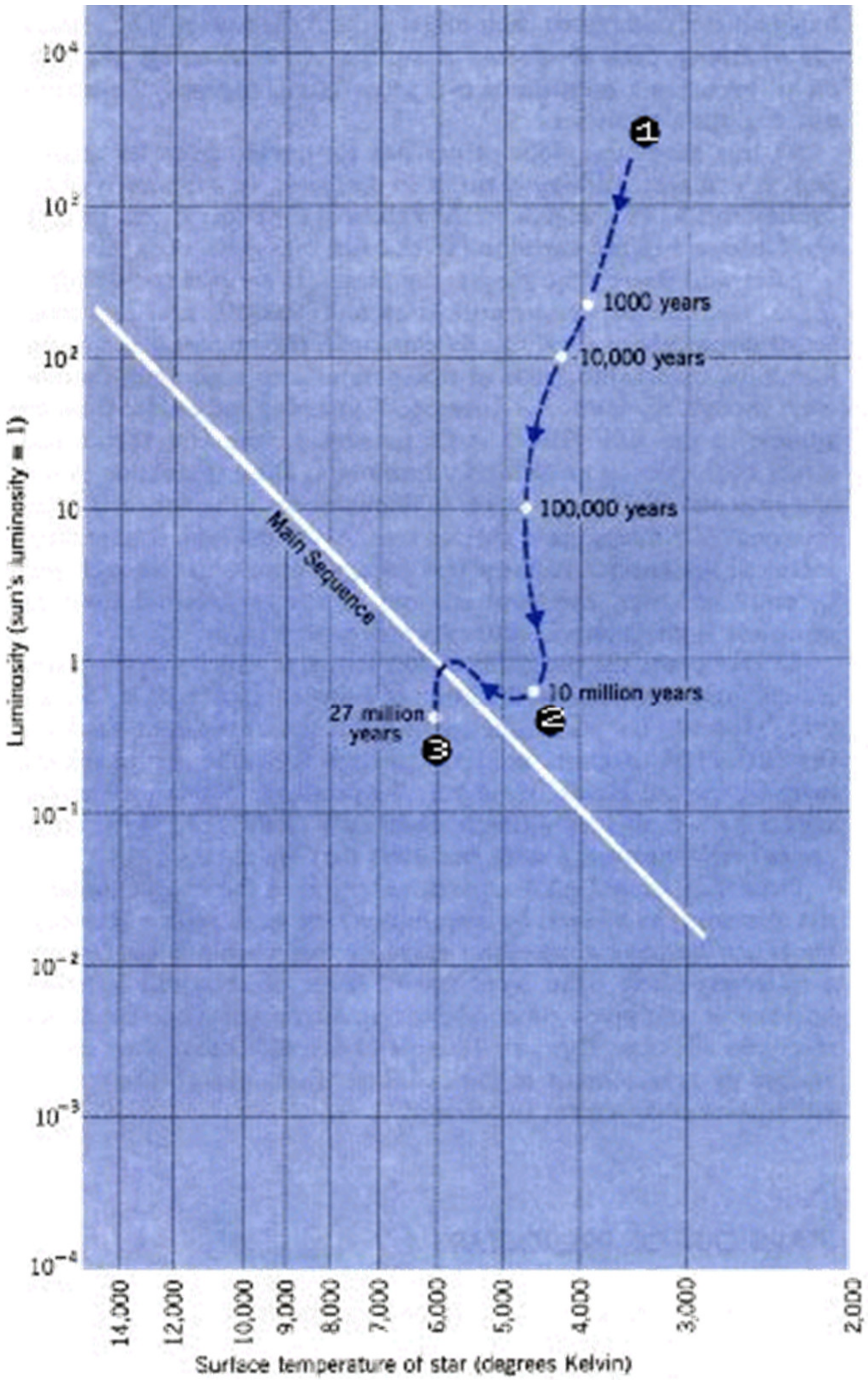
$$R = 6.96\times10^{10}\, cm$$

$$L_\odot = 3.828\times10^{33}\, erg\cdot s^{-1}$$

During this stage the interior is cool, therefore, fully convective.



Schematic picture of the evolution
of the radius and luminosity
after the birthline.



1. Protostar

The initial collapse occurs quickly, over a period of a few years. As the star heats up, pressure builds up following the perfect gas law. The outward pressure nearly balances the inward gravitational pull, a condition called hydrostatic equilibrium.

Age: 1–3 yrs; $R \sim 50 R_{\odot}$; $T_{\text{core}} = 150,000$ K; $T_{\text{surface}} = 3500$ K; Energy Source: Gravity;

The star is cool, so its color is red, but it is very large so it has a high luminosity and appears at the upper right in the H-R Diagram.

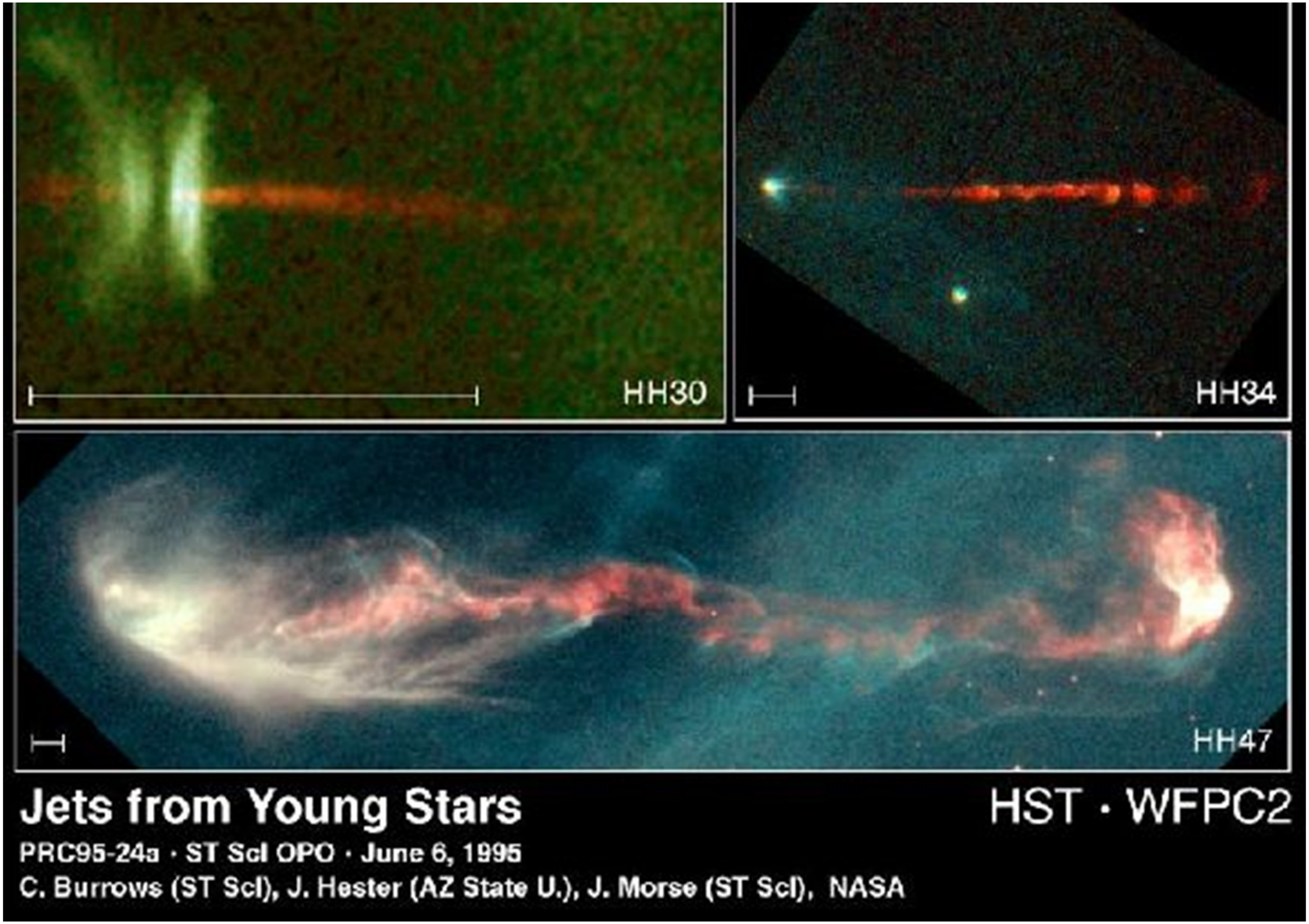
1→2. Pre-Main Sequence.

Once near-equilibrium has been established, the contraction slows down, but the star continues to radiate energy (light) and thus must continue to contract to provide gravitational energy to supply the necessary luminosity. Once nuclear reactions begin in the core, the star readjusts to account for this new energy source. During this phase, the star is above the main sequence; such pre-main sequence stars are observed as T-Tauri Stars, which are going through a phase of high activity Material is still falling inward onto the star, but the star is also spewing material outward in strong winds or jets.

Age: 10 million yrs; $R \sim 1.33 R_{\odot}$; $T_{\text{core}} = 10,000,000$ K $T_{\text{surface}} = 4500$ K Energy Source: Gravity; P-P Chain turns on.

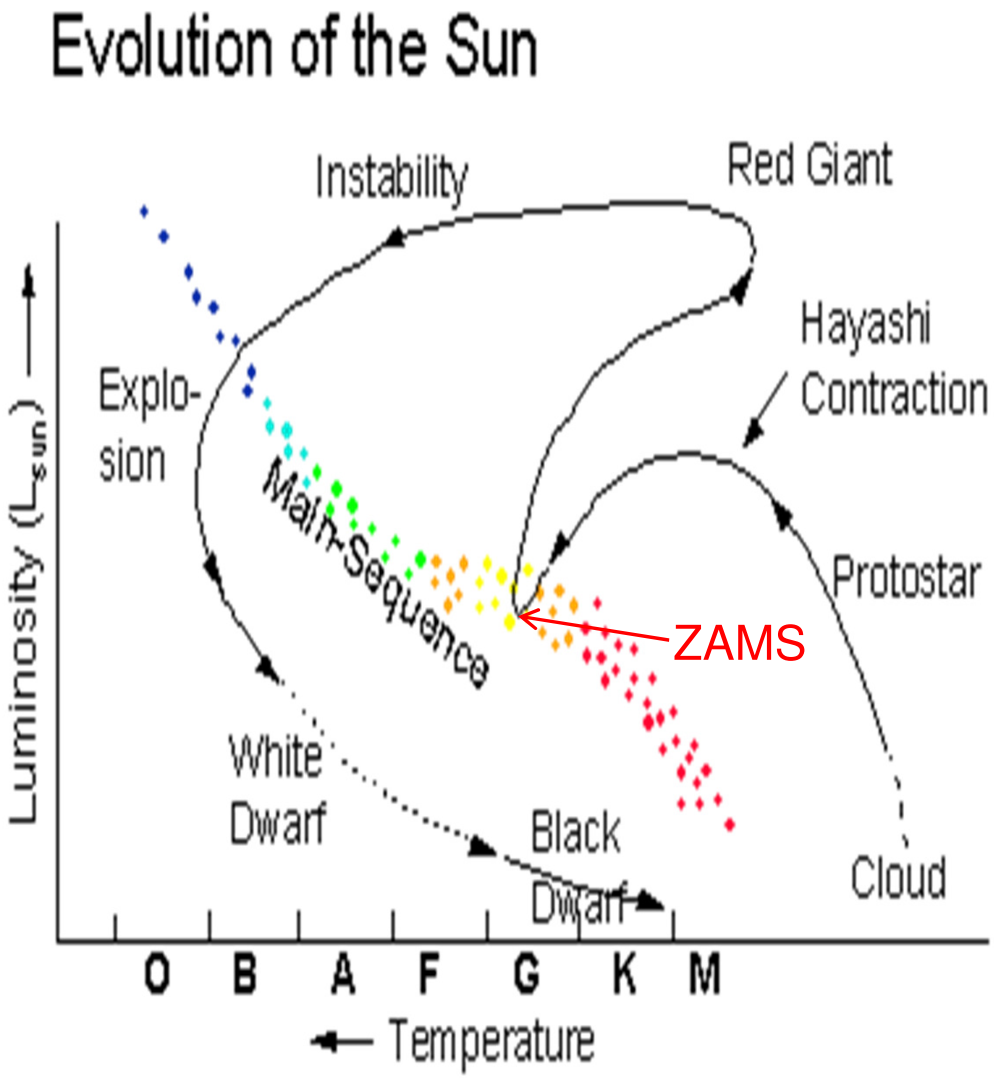
2→3. Zero Age Main Sequence.

It takes another several million years for the star to settle down on the main sequence.



Hubble Space Telescope images of young stars called T Tauri stars in the Pre-Main Sequence phase. These stars lie above the Main Sequence in the H-R Diagram. The photos show material still accreting onto the star from a protostellar disk; some of this material is ejected in high velocity jets.

Main-sequence evolution



The main sequence is not a line, but a band in the H-R Diagram. Stars start out at the lower boundary, called the Zero-Age Main Sequence (ZAMS) referring to the fact that stars in this location have just begun their main sequence phases. Because the transmutation of Hydrogen into Helium is the most efficient of the nuclear burning stages, the main sequence phase is the longest phase of a star's life, about 10 billion yrs for a star with 1 solar mass.

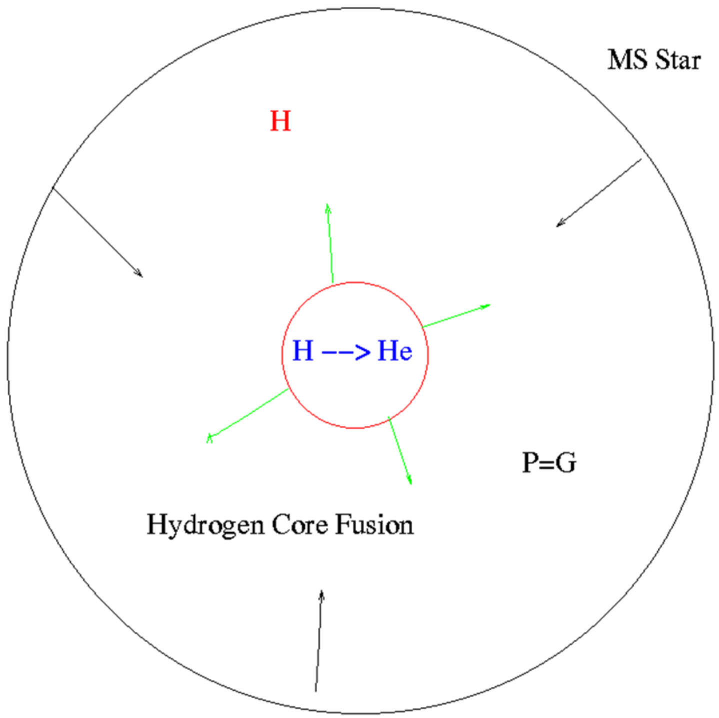
ZAMS age: 27 million yrs; $R \sim R_{\odot}$;

$T_{\text{core}} = 15{,}000{,}000K$;

$T_{\text{surface}} = 6000K$;

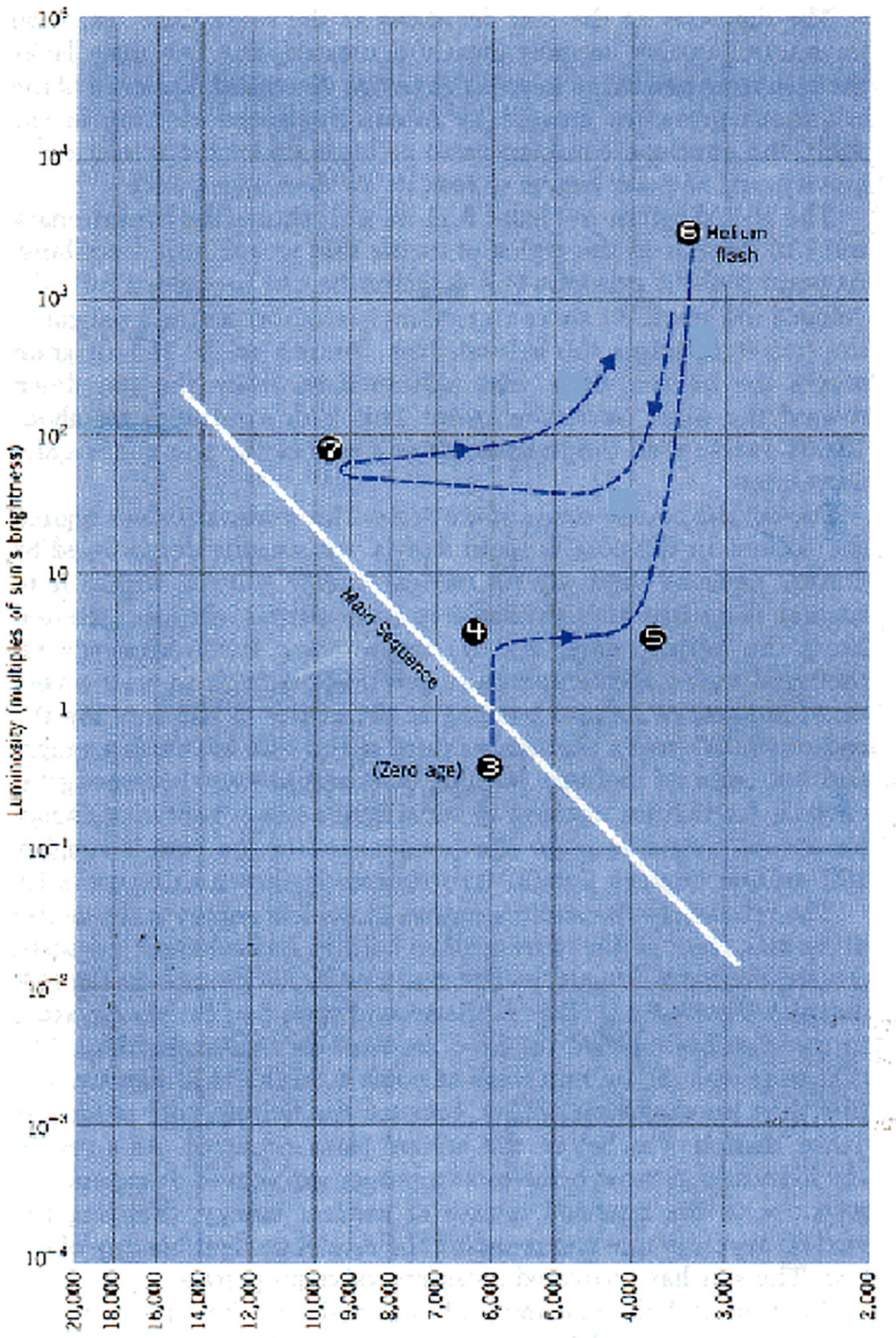
Energy Source: P-P Chain in core.

Post-Main Sequence Evolution



During the main sequence phase there is a "feedback" process that regulates the energy production in the core and maintains the star's stability. A good way to see the stability of this equilibrium is to consider what happens if we depart in small ways from equilibrium: Suppose that the amount of energy produced by nuclear reactions in the core is not sufficient to match the energy radiated away at the surface. The star will then lose energy; this can only be replenished from the star's supply of gravitational energy, thus the star will contract a bit.

As the core contracts it heats up a bit, the pressure increases, and the nuclear energy generation rate increases until it matches the energy required by the luminosity. Similarly, if the star overproduces energy in the core the excess energy will heat the core, increasing the pressure and allowing the star to do work against gravity. The core will expand and cool a bit and the nuclear energy generation rate will decrease until it once again balances the luminosity requirement of the star.



3→4. End of Main Sequence

Age: 10 billion yrs; Energy Source: P-P Chain in a shell around the core.

4→5. Post Main Sequence

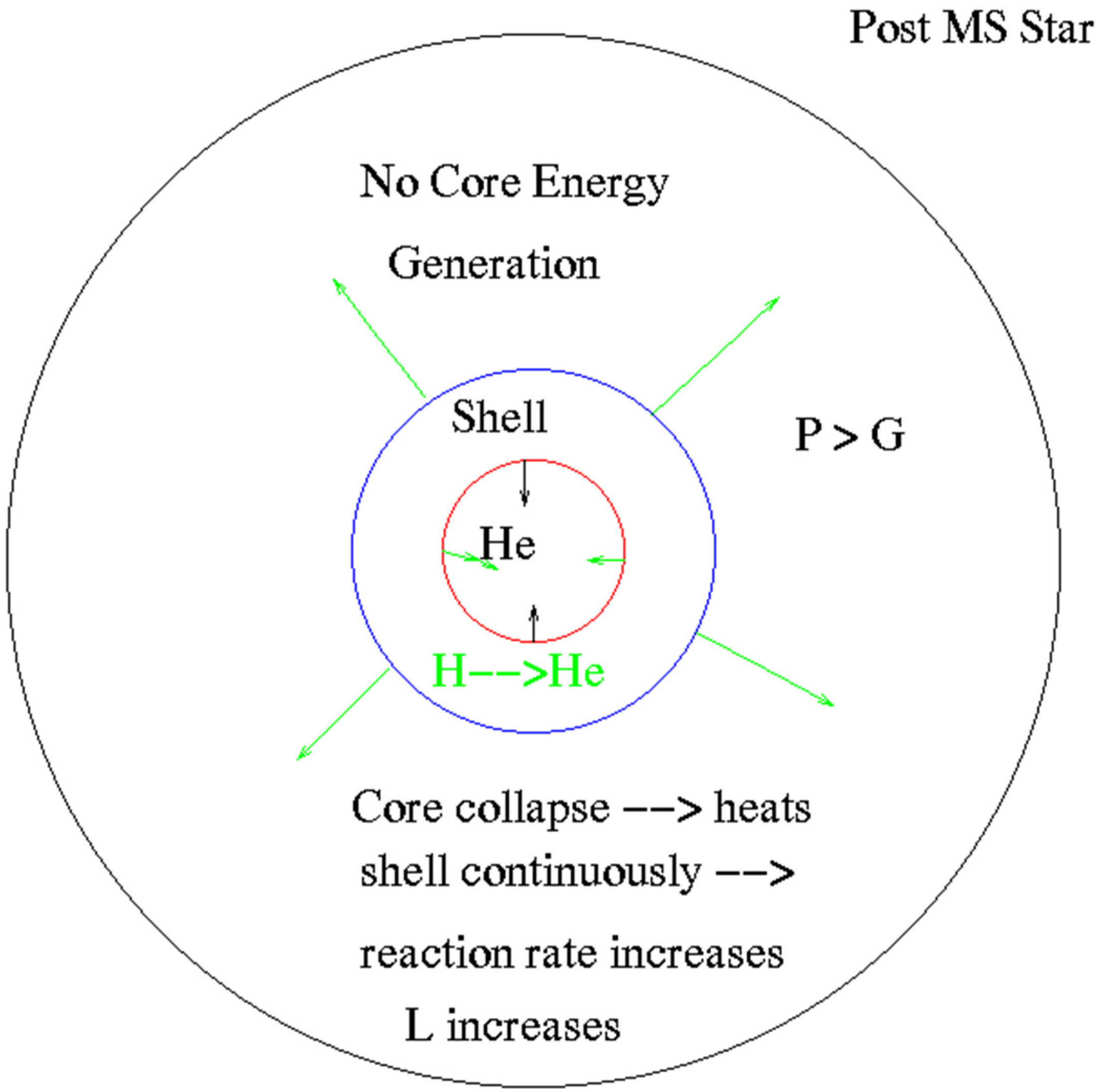
Age: About 1 billion years from Point 4; $R \sim 2.6 R_{\odot}$; $T_{\text{surface}} = 4500$ K; Energy Source: p-p chain in a shell, gravitational contraction of the core.

5→6. Red Giant - Helium Flash

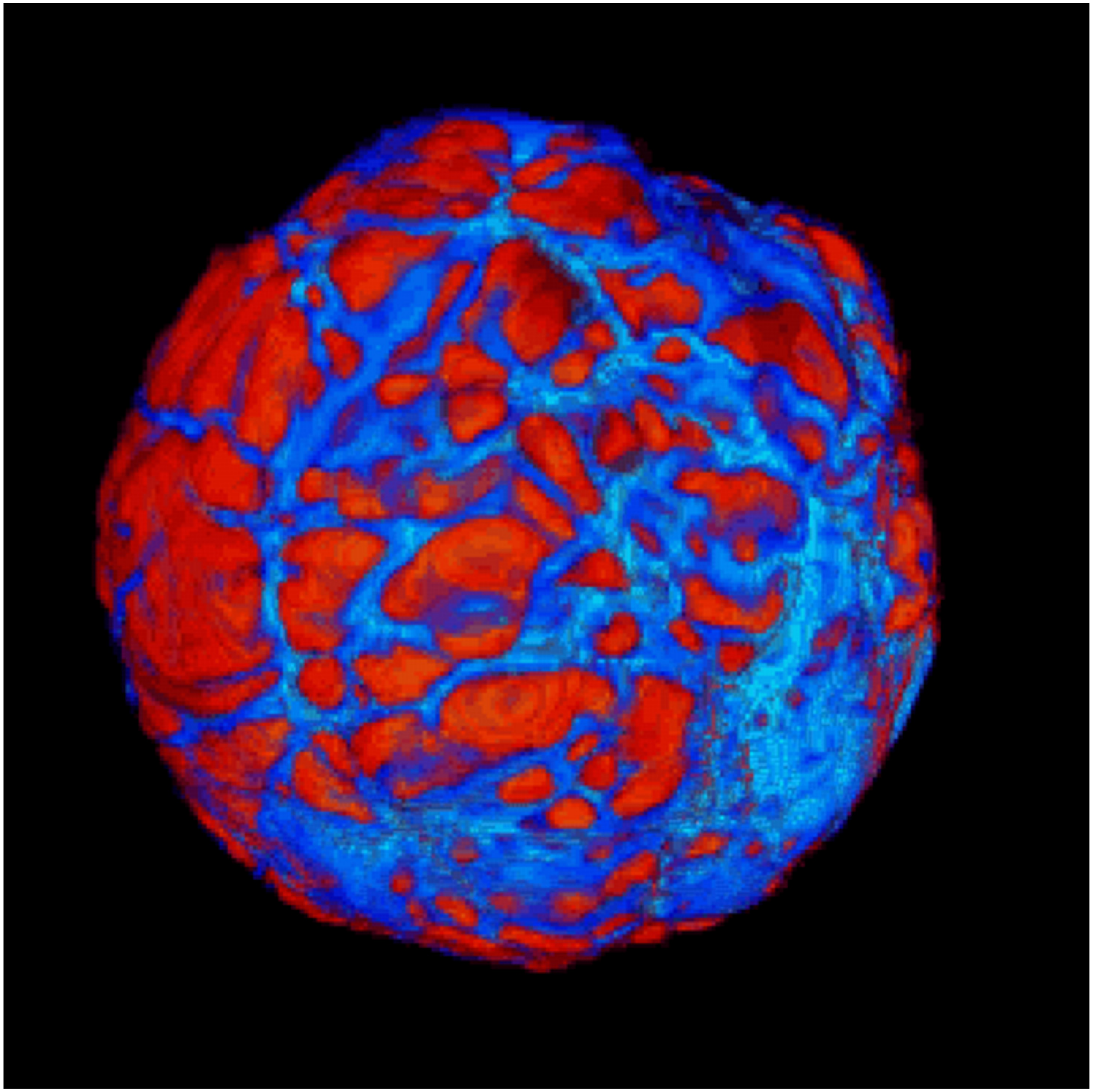
As the Helium core of the star contracts, nuclear reactions continue in a shell surrounding the core. Initially the temperature in the core is too low for fusion of helium, but the core-contraction liberates gravitational energy causing the helium core and surrounding hydrogen-burning shell to increase in temperature, which, in turn, causes an increase in the rate of nuclear reactions in the shell. This extra energy output pushes the stellar envelope outward, against the pull of gravity, causing the outer atmosphere to grow by as much as a factor of 200. The star is now cool, but very luminous - a Red Giant.

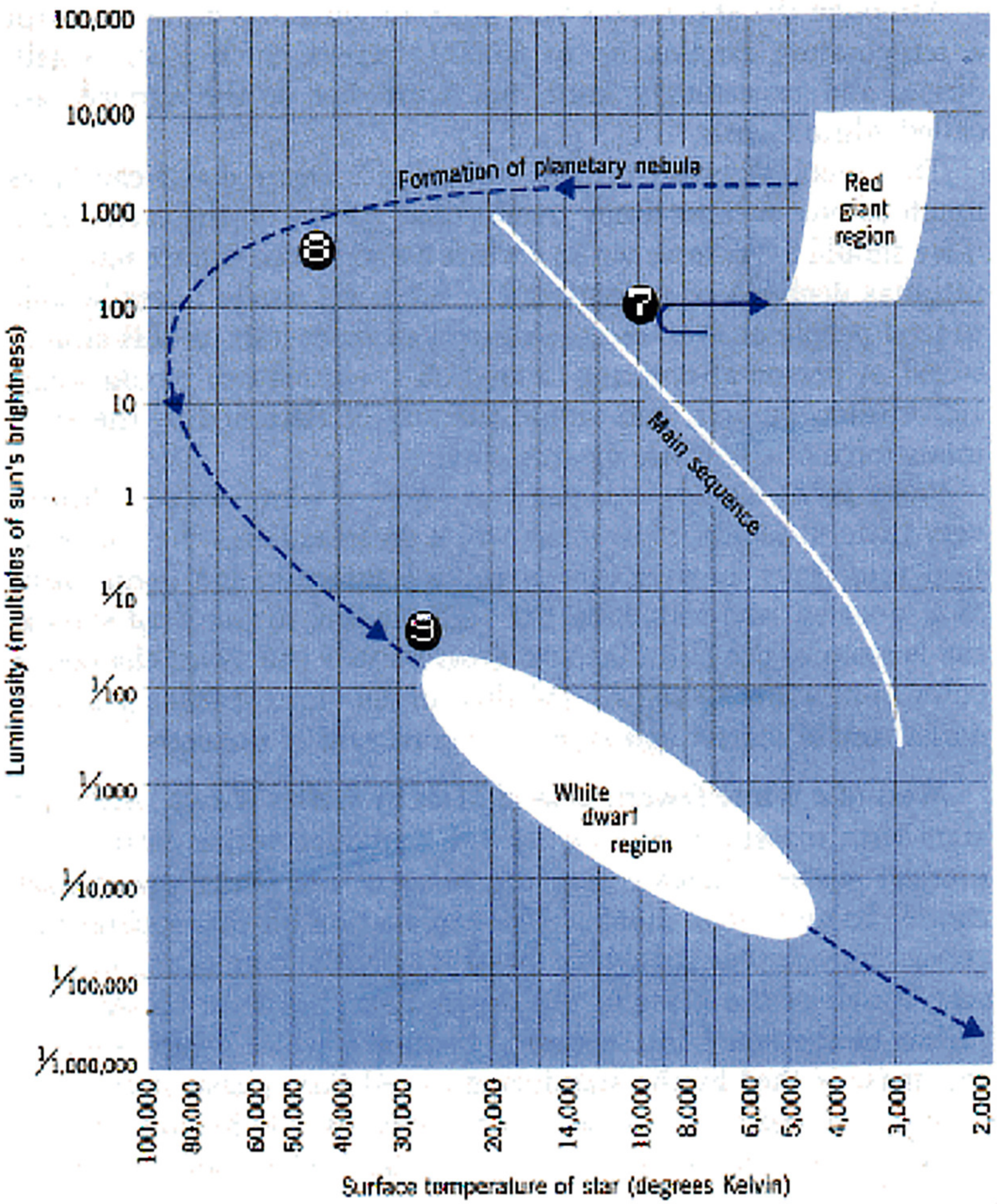
Age: 100 million yrs from Point 5; $R \sim 200 R_{\odot}$; $T_{\text{core}} = 200{,}000{,}000 K$ $T_{\text{surface}} = 3500 K$.

Post Main-Sequence Evolution



Simulating a Pulsating Red Giant Star, Porter, et al





6→7. Helium Burning Main Sequence

Once again the core of the star readjusts to allow for a new source of energy, in this case fusion of Helium to form Carbon via the Triple-Alpha Process. The Triple alpha process releases only about 20% as much energy as hydrogen burning, so the lifetime on the Helium Burning Main Sequence is only about 2 billion years.

Age: About 10,000 yrs from point 6; $T_{\text{surface}} = 9000\,\text{K}$; $T_{\text{core}} = 200{,}000{,}000\,\text{K}$; Energy Source: Triple-alpha process in the core; P-P Chain in the shell

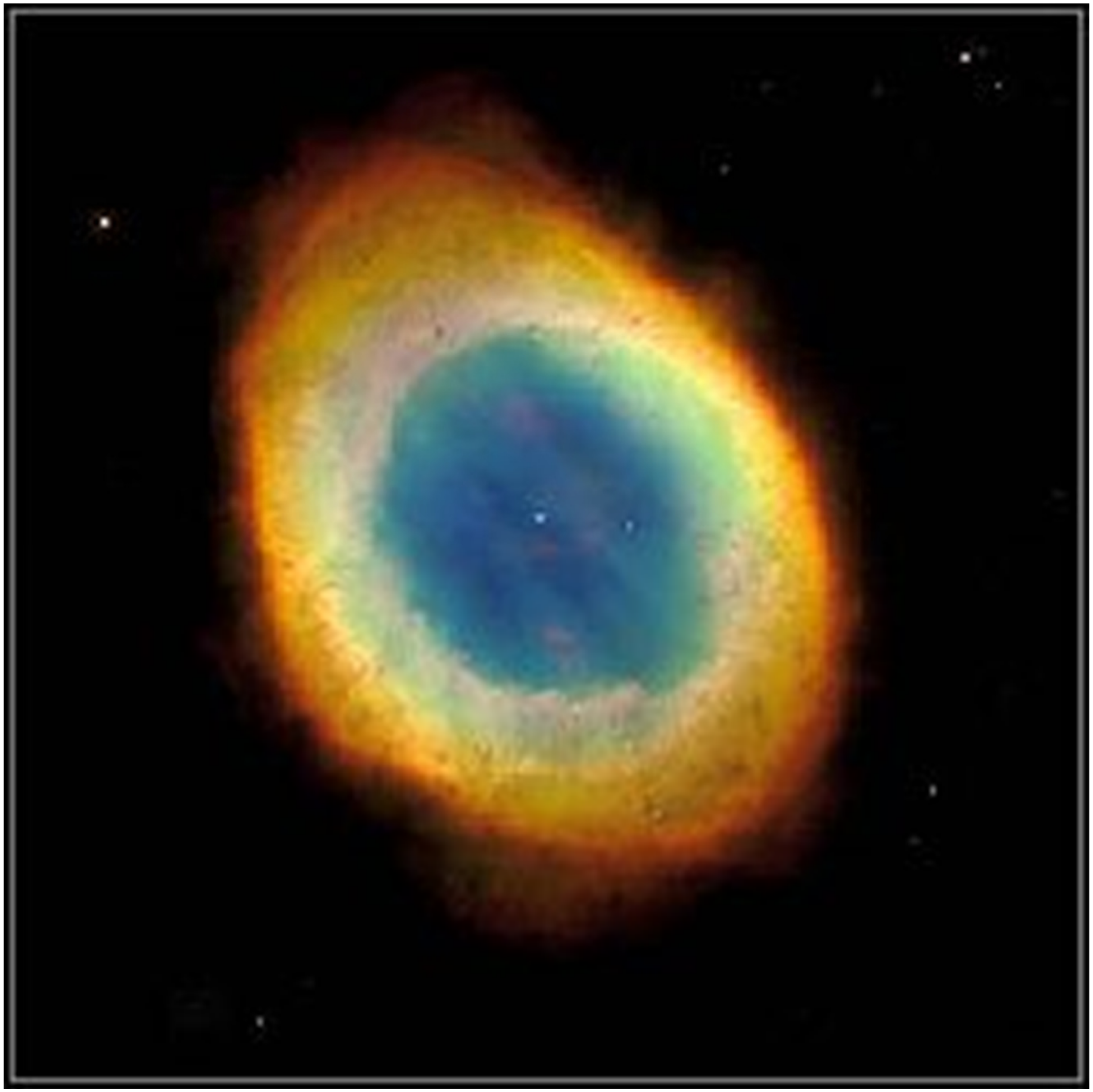
7→8. Planetary Nebula

When the helium is exhausted in the core of a star like the sun, the C-O core will begin to contract again. Throughout the star's lifetime it is losing mass via a stellar wind. During Helium Shell Burning, a final thermal pulse causes the star to eject as much of 10% of its mass, the entire outer envelope, revealing the hot inner regions with temperatures in excess 100,000K. The resulting Planetary Nebula is the interaction of the newly ejected shell of gas with the more slowly moving ejecta from previous events and the ultraviolet light from the hot stellar remnant, which heats the gas and causes it to fluoresce.

8→9. White Dwarf

As the nebula disperses, the shell nuclear reactions die out leaving the stellar remnant, supported by electron degeneracy, to fade away as it cools down. The white dwarf is small, about the size of the earth, with a density of order 1 million g/cm^3.

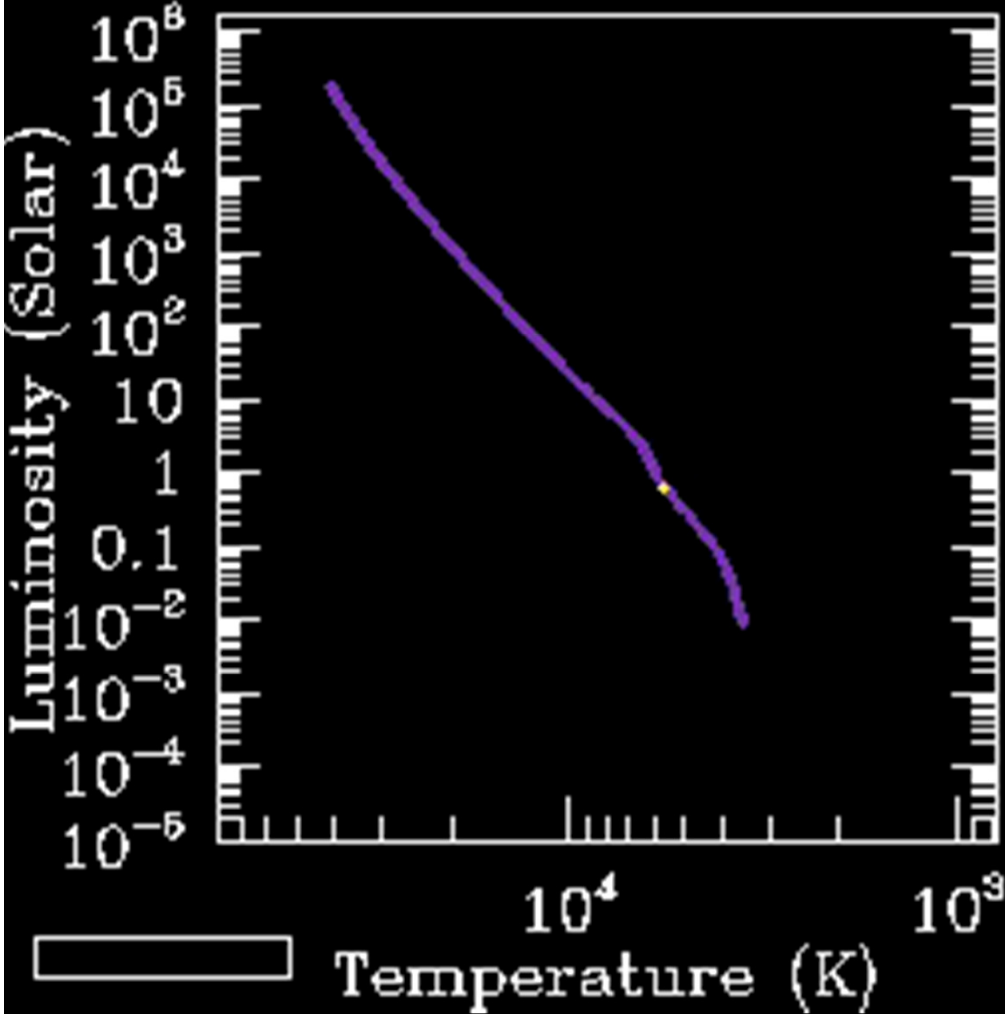
$R \sim R_{\text{earth}}$ (a few thousand km); $T_{\text{surface}} = 30000 - 5000\,\text{K}$.



The Ring Nebula in Lyra (Messier Database, Web Nebulae) is the prototypical Planetary Nebula. Rather than a spherical shell as initially believed, the Ring's shape is probably a torus or cylinder of gas, seen nearly pole-on. Its age is estimated to be a few thousand years; the central star has a surface temperature over 100,000K.

The Planetary Nebula phase is relatively short lived, estimated to be about 25,000 years, and there are about 10,000 planetaries in the Milky Way.

Sun’s Evolution on HR Diagram



Equations of Stellar Structure

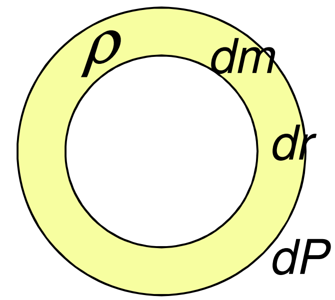
1. Hydrostatic Equations

Basic assumptions:

1. hydrostatic equilibrium: gravity force = pressure gradient;
2. thermal balance: energy generation rate = luminosity.

Consider a thin spherical shell of radius r, thickness dr, mass dm, and density ρ. The mass conservation equation is:

$$dm = 4\pi\rho r^2 dr$$



or

$$\frac{dm}{dr} = 4\pi\rho r^2 .$$

The balance between the pressure and gravity forces is:

$$4\pi r^2 dP = -\frac{Gmdm}{r^2} = -Gm\frac{4\pi\rho r^2 dr}{r^2},$$

or

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}.$$

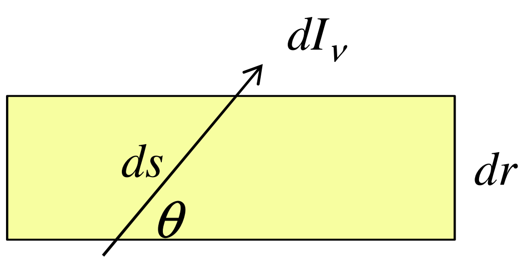
2. Radiative Energy Transfer

Energy inside a star is transported by radiation and convection. Consider the radiative transport:

$$dI_\nu(\theta) = -\kappa'_\nu I_\nu(\theta) ds + \varepsilon_\nu ds,$$

where I_ν is the radiation intensity at frequency ν in direction θ, κ'_ν is the absorption coefficient per cm of the light path, ε_ν is the emission coefficient, $ds = dr / \cos\theta$. Then,

$$\cos\theta \frac{dI_\nu}{dr} = -\kappa'_\nu I_\nu(\theta) + \varepsilon_\nu = -\kappa'_\nu \left(I_\nu - \frac{\varepsilon_\nu}{\kappa'_\nu} \right).$$



The ratio $\varepsilon_\nu / \kappa'_\nu$ is called source function S_ν.

Kirchoff's law: in Local Thermodynamic Equilibrium (LTE) every transition is balanced by reversed, and

$$S_\nu \equiv \frac{\varepsilon_\nu}{\kappa'_\nu} = B_\nu(T)$$

is a function of temperature, $B_\nu(T)$ - Plank function.

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}.$$

If $I_\nu = B_\nu$ then radiation is isotropic, and there is no energy transfer.

For the non-isotropic LTE case the radiative transfer equation

$$\cos\theta \frac{dI_\nu}{dr} = -\kappa'_\nu \left(I_\nu - B_\nu\right).$$

can be solved by iterations assuming that (I_ν - B_ν) is small. From the first iteration:

$$\cos\theta \frac{dB_\nu}{dr} = -\kappa'_\nu \left(I_\nu - B_\nu\right).$$

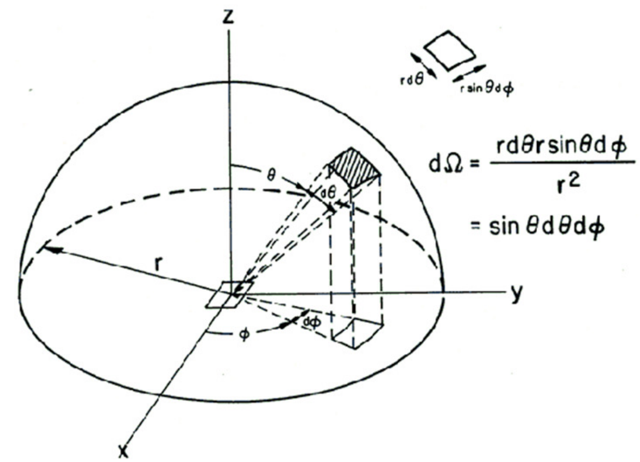
we find:

$$I_\nu = B_\nu - \frac{\cos\theta}{\kappa'_\nu} \frac{dB_\nu}{dr}$$

or using the absorption coefficient per unit mass: $\kappa_\nu = \kappa'_\nu \; / \; \rho$

$$I_\nu = B_\nu - \frac{\cos\theta}{\rho\kappa_\nu} \frac{dB_\nu}{dr}.$$

Then, the energy flux for frequency ν, F_ν, is integral of $I_\nu \cos\theta$ over a hemisphere:



$$F_\nu = \int_\Omega I_\nu \cos\theta d\Omega = -2\pi \frac{\int_0^1 \cos^2\theta d\cos\theta}{\kappa_\nu \rho} \frac{dB_\nu}{dr} = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{dB_\nu}{dr}$$

Integrating over frequency ν we get the total energy flux, F:

$$F = \int_0^\infty F_\nu d\nu = -\int_0^\infty \frac{4\pi}{3\kappa_\nu \rho}\frac{dB_\nu}{dr} d\nu \approx -\frac{4\pi}{3\kappa\rho}\int_0^\infty \frac{dB_\nu}{dr} d\nu,$$

where $$\frac{1}{\kappa} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$$

is the Rosseland mean absorption coefficient, or ‘opacity’ coefficient.

Integrating over frequency ν we get the total energy flux, F :

$$F = \int_0^\infty F_\nu d\nu = -\int_0^\infty \frac{4\pi}{3\kappa_\nu \rho} \frac{dB_\nu}{dr} d\nu \approx -\frac{4\pi}{3\kappa\rho} \int_0^\infty \frac{dB_\nu}{dr} d\nu,$$

where $$\frac{1}{\kappa} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$$

is the Rosseland mean absorption coefficient, or 'opacity' coefficient.

Now, calculate $$\int_0^\infty B_\nu d\nu = \int_0^\infty \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu = \frac{ac}{4\pi} T^4,$$

where $$a = \frac{8\pi^5 k^4}{15 c^3 h^3}.$$

Then, $$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}.$$

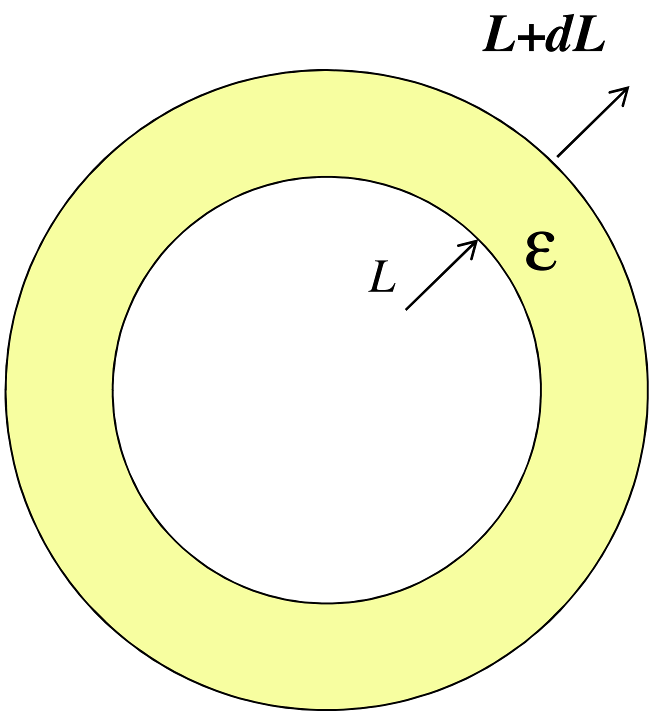
The total energy flux, $L = 4\pi r^2 F$, integrated over a sphere of radius r :

$$L = -\frac{16\pi acT^3}{3\kappa\rho} \frac{dT}{dr}.$$

Energy transfer and balance equations

The total energy flux, $L = 4\pi r^2 F$, integrated over a sphere of radius r:

$$L = -\frac{16\pi acT^3}{3\kappa\rho}\frac{dT}{dr}.$$



If ε is the energy release per unit mass then the energy flux change in a shell *dr* is:

$$dL = \varepsilon\rho 4\pi r^2 dr$$

$$\frac{dL}{dr} = 4\pi\rho r^2 \varepsilon$$

This is the equation for conservation of energy (energy balance).

3. Equation of State

The pressure in the solar interior can be described by the ideal gas law: $P = nkT$,

where k is the Boltzman constant and n is the particle density.

$$n = n_H + n_{He} + n_Z + n_e,$$

where n_Z is the particle density of atom heavier than helium.

The particle density can be expressed in terms of fractional mass abundances of hydrogen, X, helium, Y, and heavier elements, Z, such as

$$X + Y + Z = 1.$$

Then, $$n_H = \frac{\rho X}{M}, \quad n_{He} = \frac{\rho Y}{4M}, \quad n_Z = \frac{\rho Z}{AM}$$

where M is the proton mass, A is a mean mass of the heavy elements (typically, $A \simeq 16$).

For fully ionized plasma (in the deep interior of Sun):

$$n_e = n_H + 2n_{He} + \frac{1}{2} A n_Z$$

Then

$$n = 2n_H + 3n_{He} + (1 + \frac{1}{2} A) n_Z = \frac{\rho}{M}\left(2X + \frac{3}{4} Y + \frac{1 + A/2}{A} Z \right)$$

or $$n \approx \frac{\rho}{M}\left(2X + \frac{3}{4} Y + \frac{1}{2} Z \right).$$

Finally, $$P = nkT = \frac{k}{M} \rho T \left(2X + \frac{3}{4} Y + \frac{1}{2} Z \right) = \frac{R \rho T}{\mu},$$

where μ is the mean molecular weight:

$$\mu = \frac{1}{2X + \frac{3}{4} Y + \frac{1}{2} Z}.$$

Estimate importance of the electrostatic force.

The mean electrostatic energy is: $E_e \sim \frac{e^2}{r}$.

The mean thermal energy: $E_T \sim \frac{3}{2} kT$.

In the Sun, $\frac{E_e}{E_T} \leq 0.1$.

The electrostatic energy is small but not negligible.

Estimate the electrostatic correction to the total energy.

A charged particle creates a 'cloud' of particles of the opposite charge, and thus the electrostatic integration is limited to short distances. The Debye-Hückel theory assumes that in the neighborhood of an ion with charge e the particles are distributed according to a Boltzmann distribution:

$$n_i = ne^{-\frac{eU}{kT}} \qquad n_e = ne^{\frac{eU}{kT}},$$

where U is a mean potential around this ion. This potential is determined from Poisson equation:

$$\Delta U = -4\pi\rho_e = 4\pi e(n_i - n_e) = 4\pi en\left(e^{\frac{eU}{kT}} - e^{\frac{-eU}{kT}}\right)$$

$$\Delta U \approx \frac{4\pi ne^2 U}{kT} \equiv \frac{2U}{D^2},$$

where $D = \sqrt{\frac{kT}{8\pi n_e e^2}}$ is the Debye radius.

The solution to the Poisson equation is: $U = \frac{e}{r}e^{-\frac{r}{D}}$.

Then the correction to the ion energy due the interaction with the 'cloud' of other particles is: $\delta E = -\frac{e^2}{D}$.