

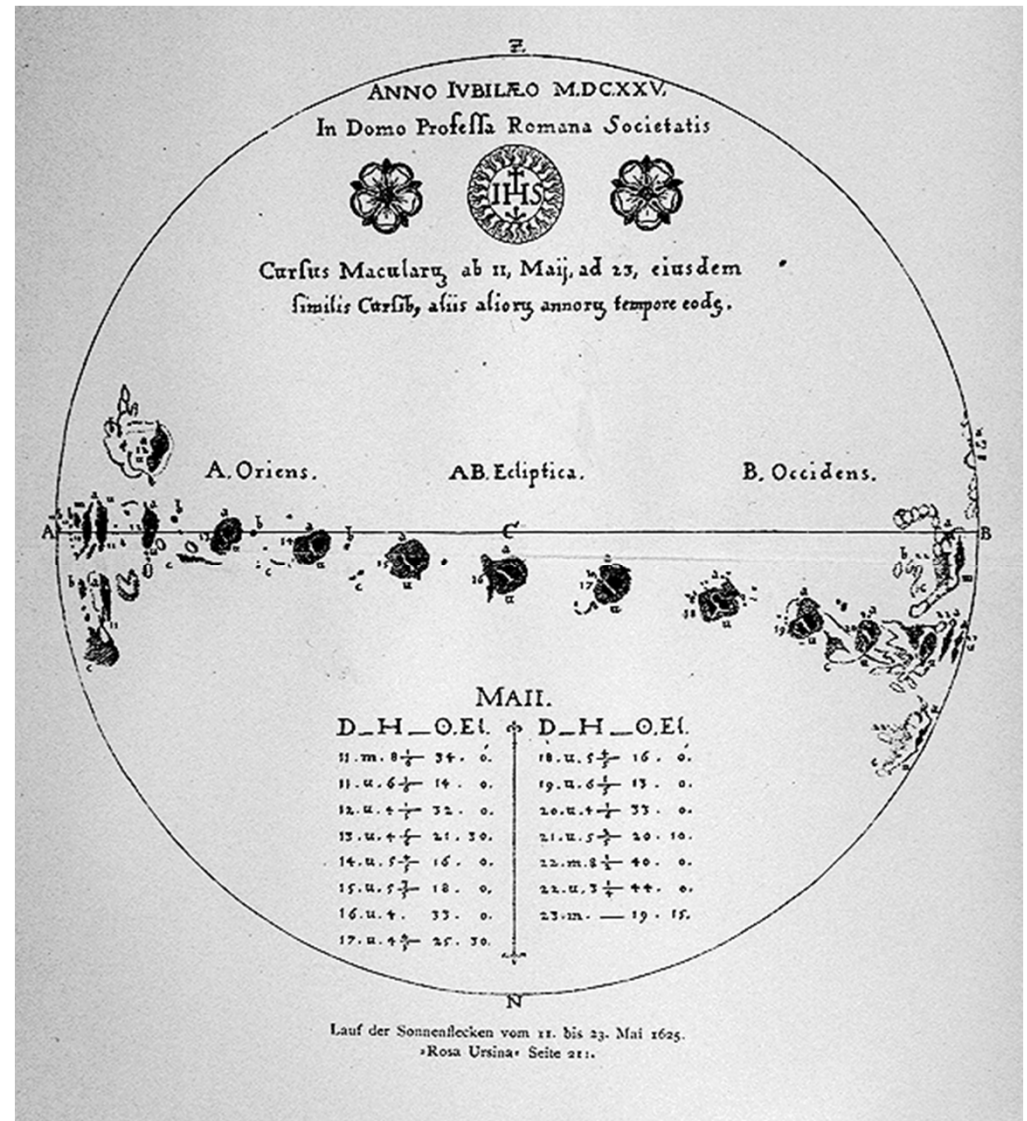
# **13. Solar Rotation**

# Solar Rotation

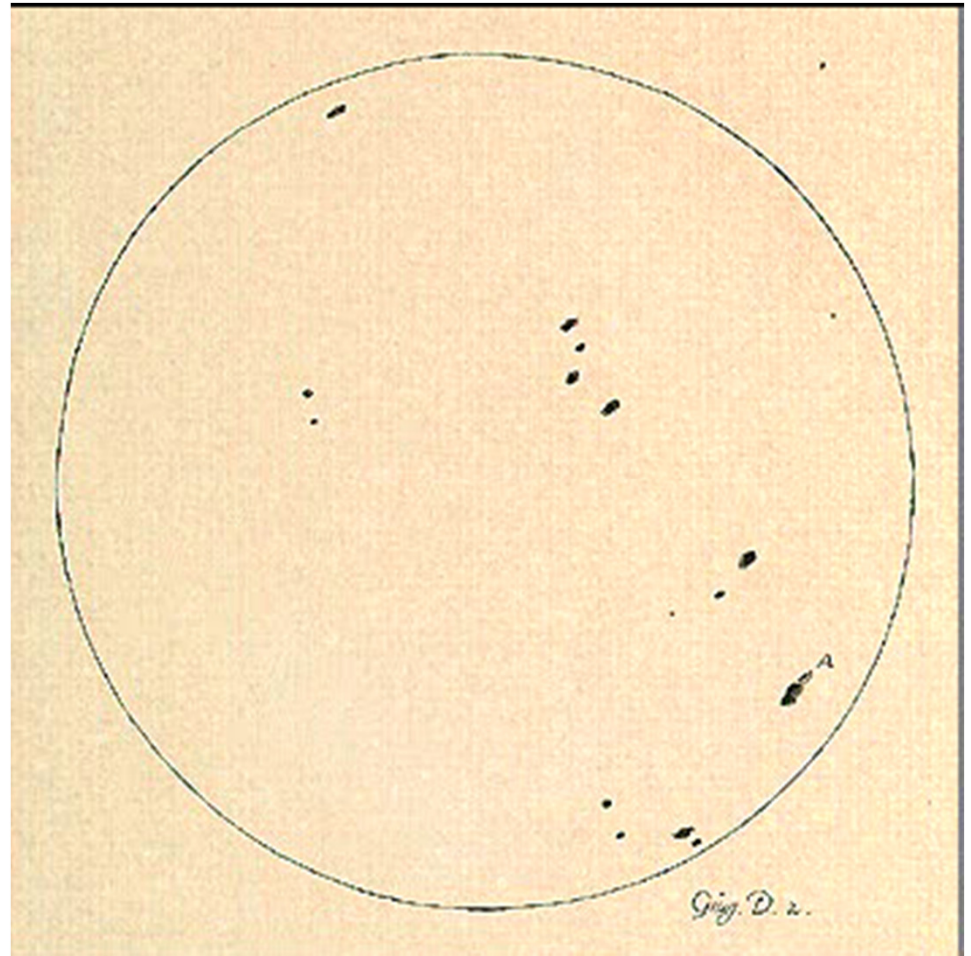
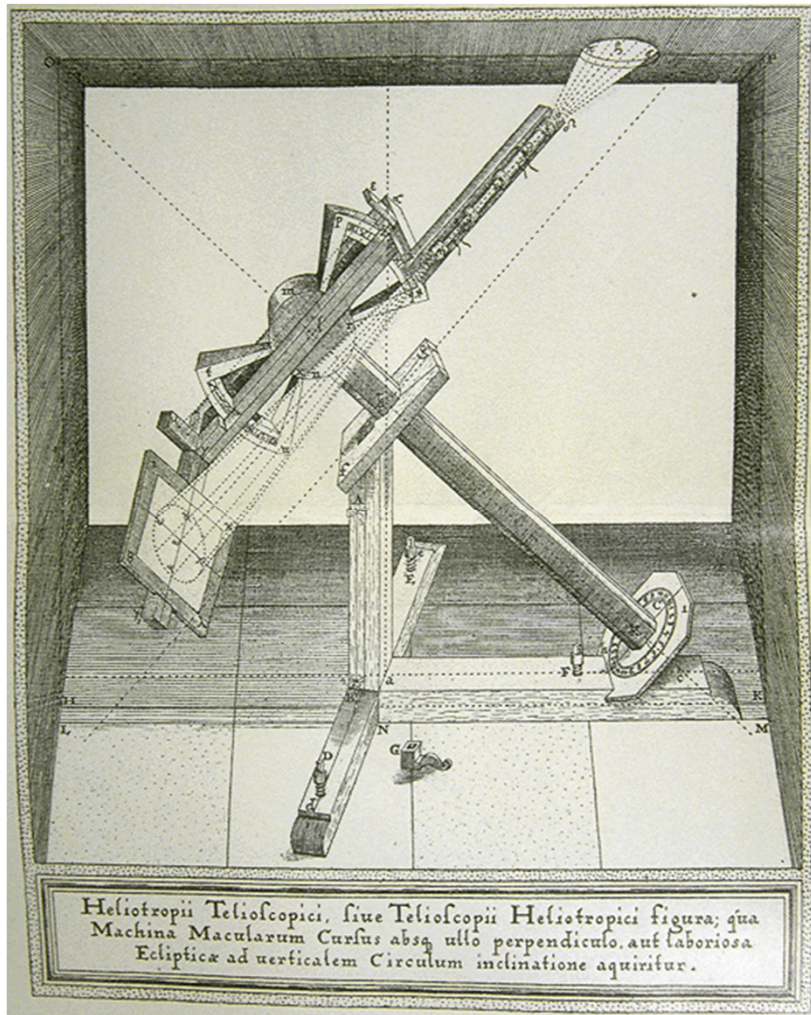
1. Observations of solar rotation
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4. Helioseismic measurements
5. Rotational frequency splitting
6. Results of helioseismic measurements
7. North-South asymmetry
8. Evolution of solar rotation
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# Observations of solar rotation

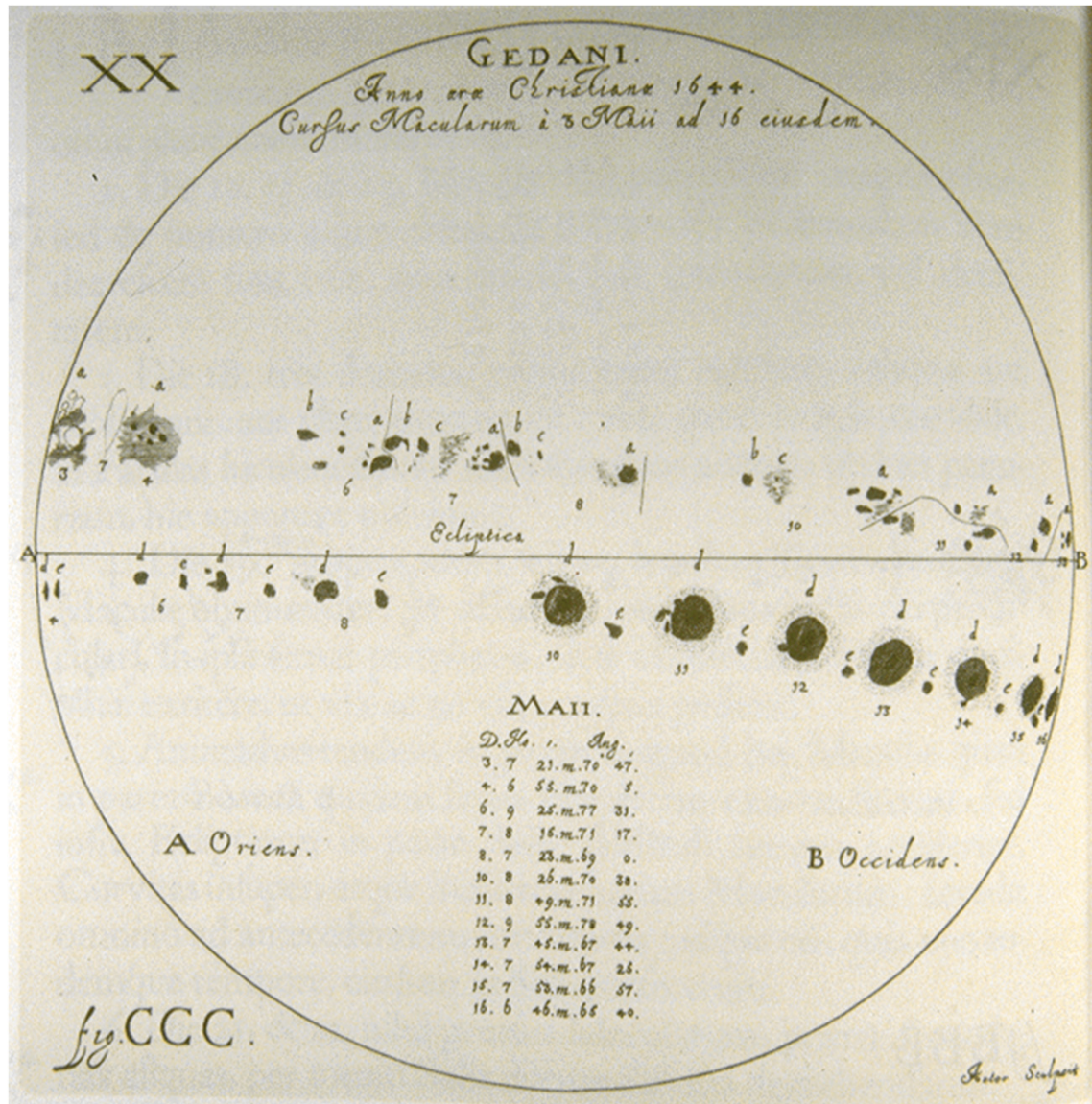
It has been known since the first observations of solar rotation around 1610 by Galileo Galilei, Johannes Fabricius and Christoph Scheiner that the solar equator rotates faster than the regions closer to the poles (latitudinal differential rotation). However, the accurate determination of the differential rotation on the solar surface and in the deep interior, is still one of the major problems of solar physics. A unique tool for determining the rotation of the Sun's interior is provided by helioseismology.



# Galileo drawings of sunspots (1613)



# Tracer measurements



The tracer measurements are carried out either by tracking the rotation rate of individual resolved features on the surface (such as sunspots, active regions, supergranular cells etc) as they pass across the disk, or by cross-correlating magnetograms and Dopplergrams obtained at different times.

Hevelius, 1644

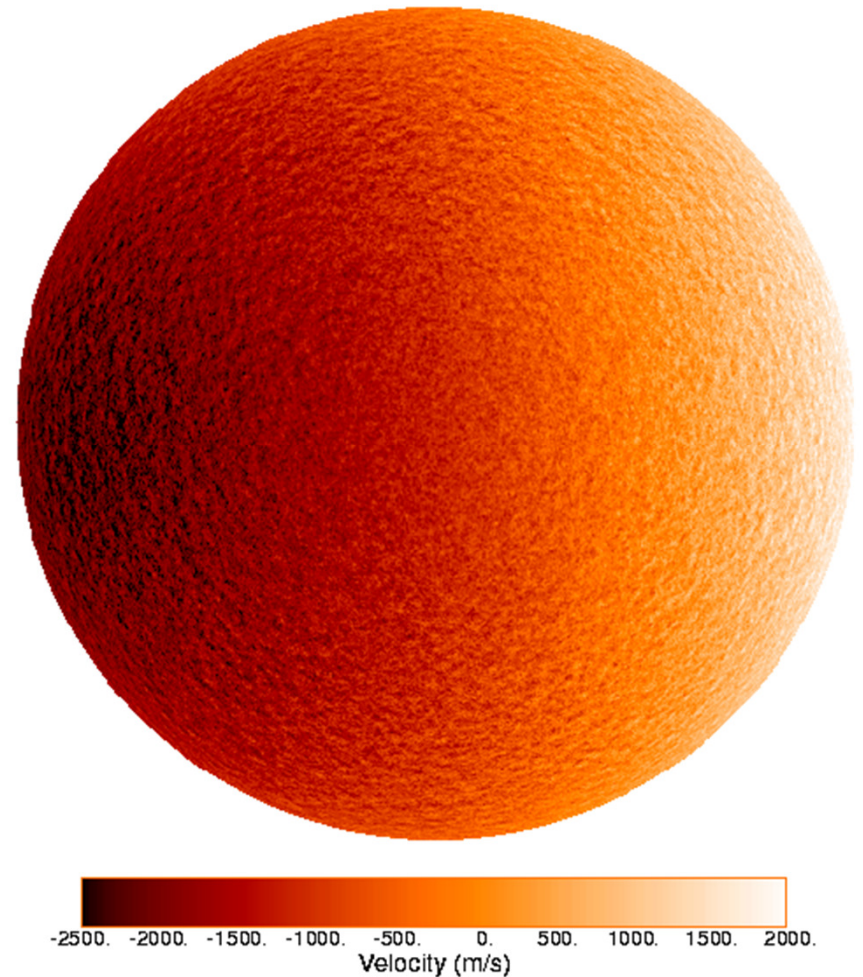
# Spectroscopic measurements

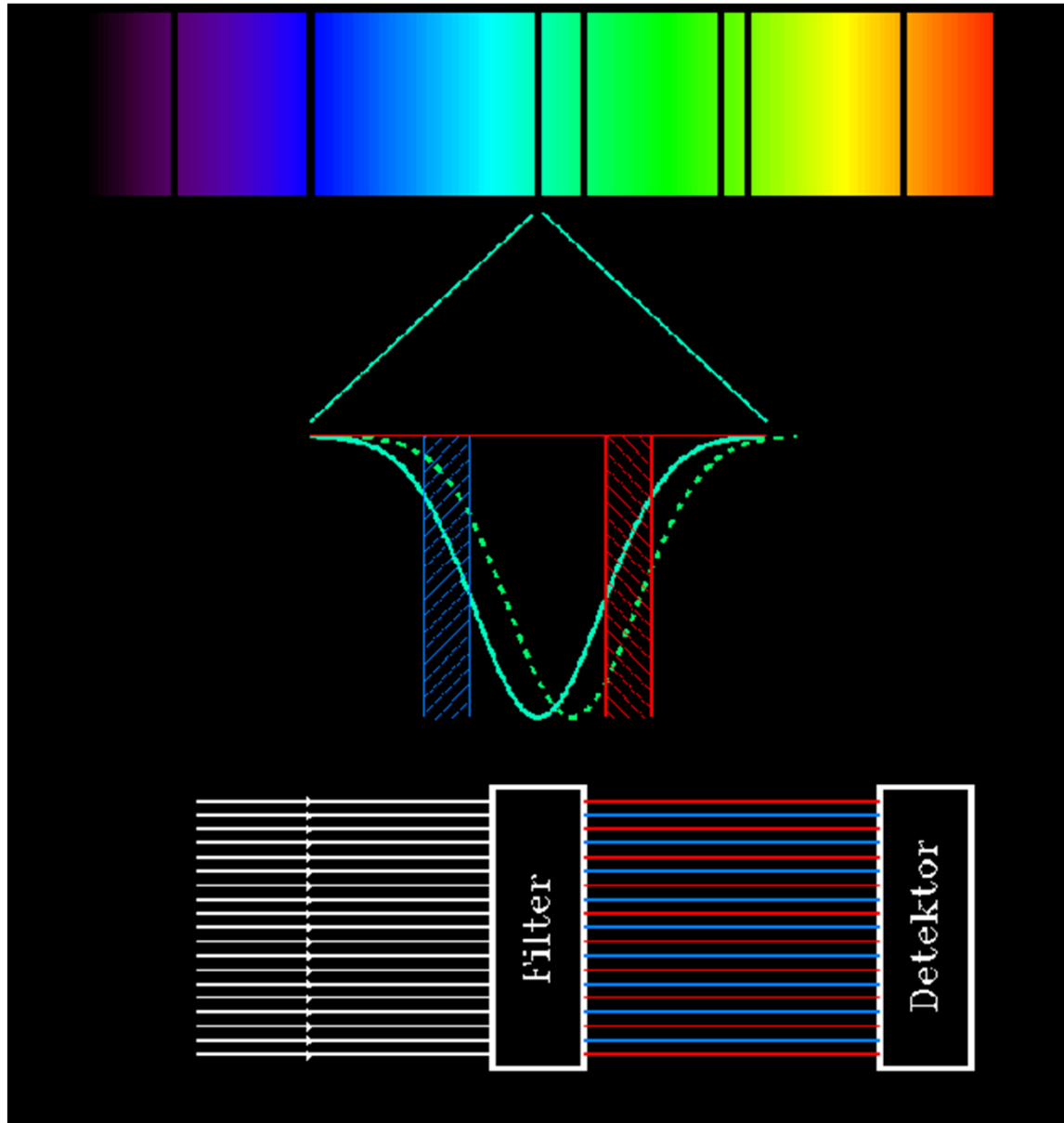
There are two basic approaches for measuring the surface rotation: spectroscopic and tracer motion.

The spectroscopic method is based on measuring the Doppler line shift. It provides the angular velocity in those layers of the solar atmosphere where the spectral lines chosen for the measurements are formed. In principle, by observing the spectroscopic shift in several lines it is possible to determine the variation of solar rotation with height in the solar atmosphere because different spectral lines are formed in different layers of the atmosphere. Such measurements of the radial variation of the rotation in the surface layers are not particularly accurate, because of the significant noise caused by large-scale convective motions - supergranulation. Nevertheless, these measurements have provided evidence that the lower layers of the solar atmosphere, and thus presumably the subsurface layers, rotate faster. This was later confirmed by helioseismology.

Single Dopplergram

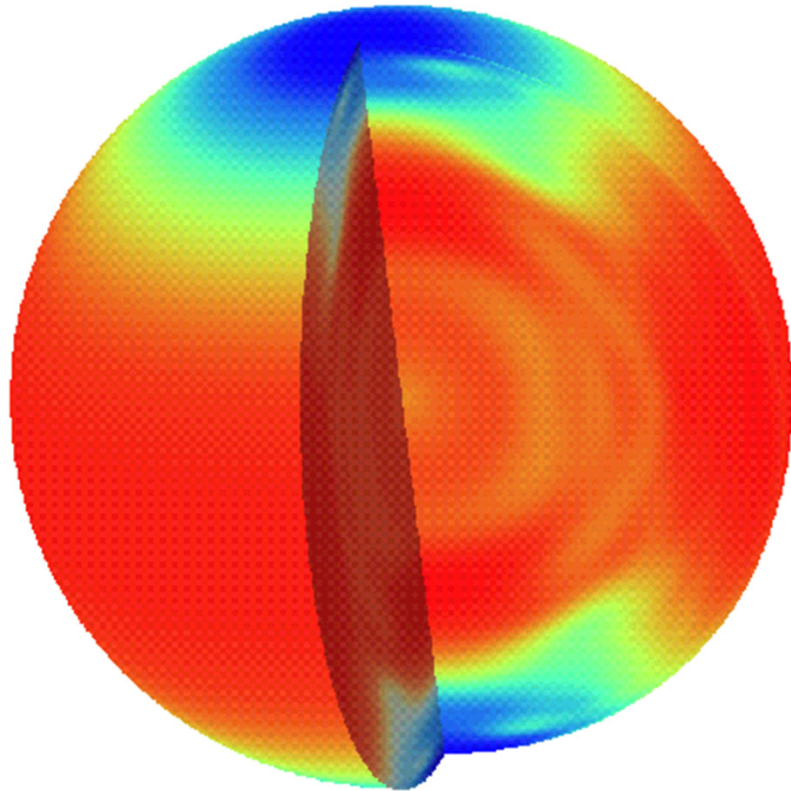
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Doppler observations of velocities on the Solar surface

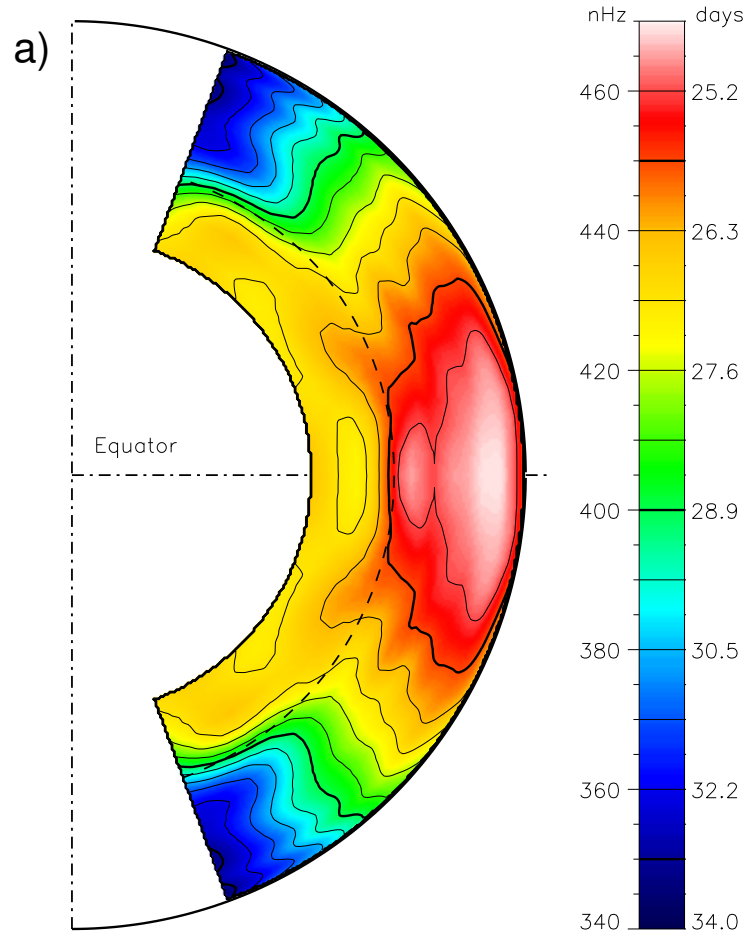
# Helioseismology measurements of internal rotation



Solar rotation rate inside the Sun measured by helioseismology

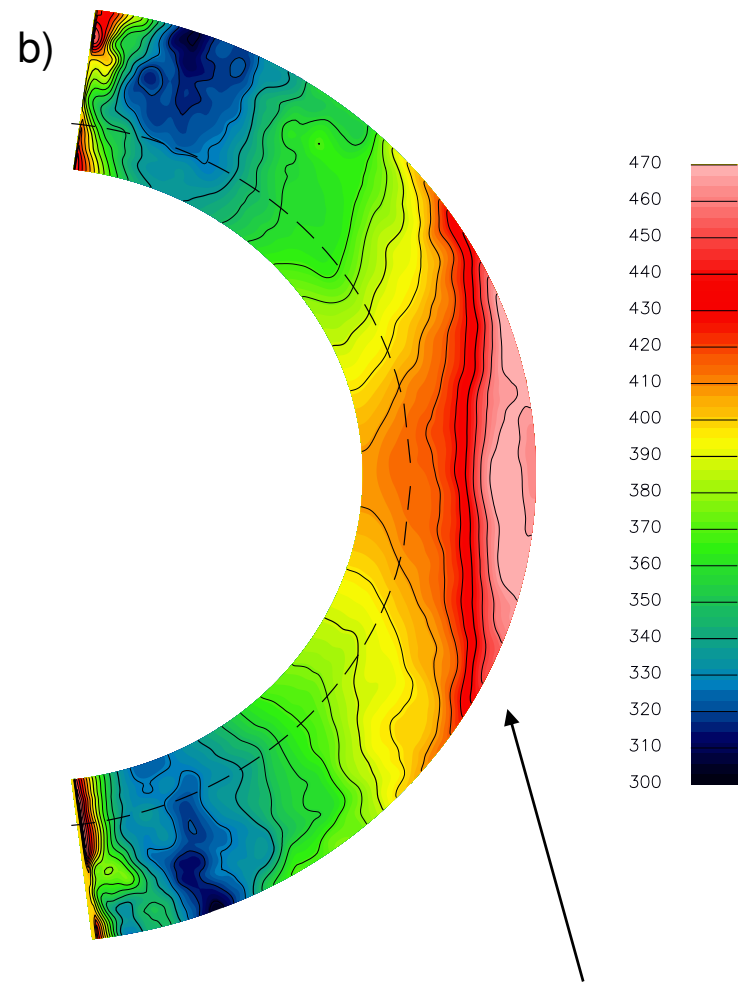
The Sun rotates differentially, i.e. the solar rotation rate varies with both latitude and radius. The differential rotation is particularly prominent in the convection zone where equatorial zones rotate almost 30% faster than near-polar regions. There are also strong variations of the rotation rate with radius near the surface and at the bottom of the convection zone (rotational shear layers). Contrary to the convection zone, the radiative zone rotates almost uniformly, though the rotation rate of the inner solar core has not been determined reliably. The mechanisms of the differential rotation, which are likely due to the interaction between convection and rotation, are not fully understood. The differential rotation is believed to play a major role in the dynamo mechanism of solar activity. It also affects the internal structure of the Sun because of rotational instabilities related to rotational shear layers. The knowledge of the internal rotation (in particular, whether the inner core rotates fast) is important for determining the quadrupole gravitational moment of the Sun and testing Einstein's theory of relativity.

# Helioseismology Observations



Numerical simulations has not satisfactory reproduced the observed rotation law

# Numerical Simulations



Rotation rate is constant on cylinders

## Characteristics of the surface rotation

Angular velocity  $\Omega$  of the surface differential rotation as a function of co-latitude  $\theta$  is usually represented in a parametric form. Historically, it was given in terms of trigonometric functions:

$$\Omega(\theta) = A + B \cos^2 \theta + C \cos^4 \theta,$$

where coefficient  $A$  gives the equatorial rotation rate ( $\theta=90$  deg), and coefficients  $B$  and  $C$  give the differential rotation. However, the basis functions in this expansion are not orthogonal. This results in correlated errors ('cross-talk') among the coefficients. Therefore, more recently a presentation in terms of orthogonal polynomials is used:

$$\Omega(\theta) = \bar{A} + \bar{B}T_2^1(\theta) + \bar{C}T_4^1(\theta),$$

where  $T_{2k}^1(\theta) \equiv a_k P_{2k+1}^1(\theta) / \sin \theta$ ,  $a_k = k!2^k / (2k+1)!!$ , are the Gegenbauer polynomials and  $P_\lambda^1(\theta)$  are the associate Legendre functions of order 1 and degree  $\lambda$ . For the first two terms:

$$T_2^1(\theta) = 5 \cos^2 \theta - 1, \quad T_4^1(\theta) = 21 \cos^4 \theta - 14 \cos^2 \theta + 1;$$

$a_0 = 1$ ,  $a_1 = 2/3$ ,  $a_2 = 8/15$ . Coefficients  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  are linear combinations of  $A$ ,  $B$ ,

and  $C$ :

$$\bar{A} = A + \frac{1}{5}B + \frac{3}{35}C, \quad \bar{B} = \frac{1}{5}B + \frac{2}{15}C, \quad \bar{C} = \frac{2}{21}C.$$

The equatorial angular velocity in this formulation is given by  $\bar{A} - \bar{B} + \bar{C}$ . The coefficients of the orthogonal representation are usually obtained with lower noise.

## Measurements and units of the solar rotation rate and rotational velocity

The surface angular velocity is usually measured in  $\text{deg day}^{-1}$  or  $\mu\text{rad s}^{-1}$ .

However, for internal rotation obtained by helioseismology the typical measure is the rotation rate,  $\Omega/2\pi$ , expressed in nanoHertz (nHz).

The rotation rate can be converted into the angular velocity units using the following relations:

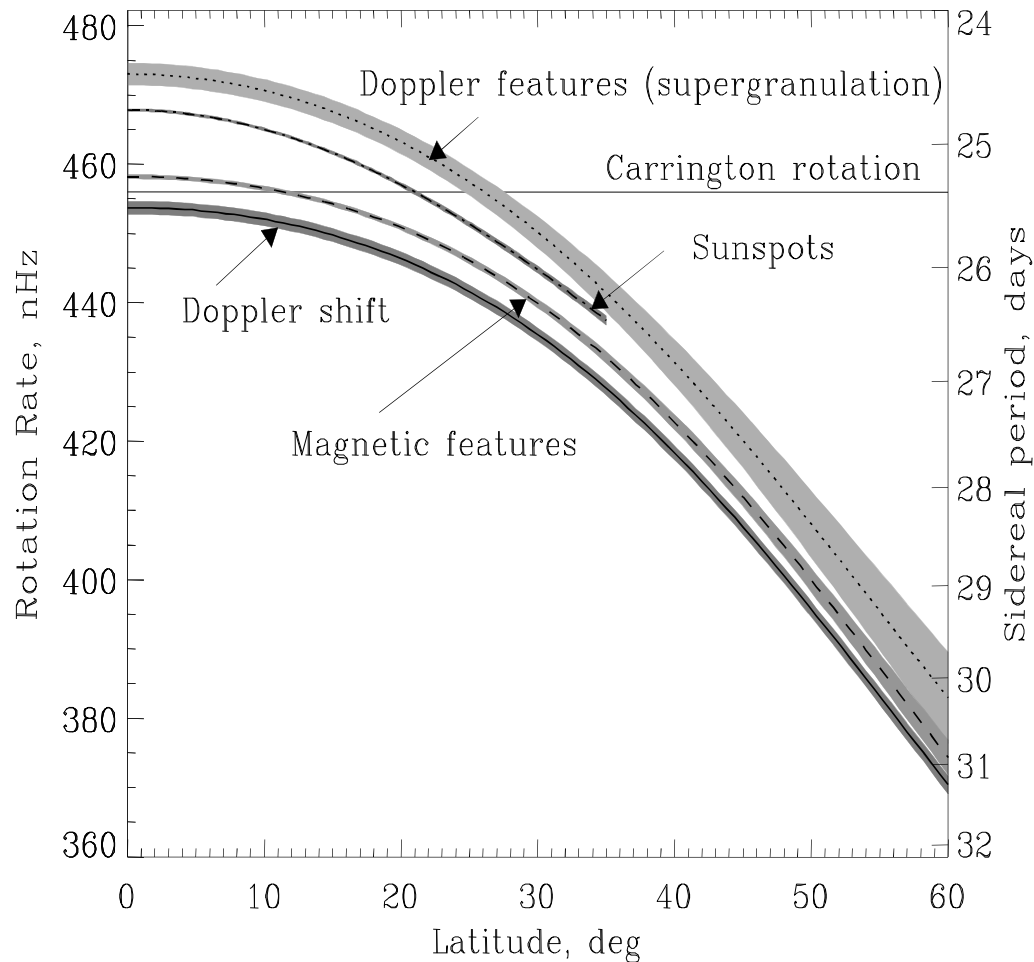
$$1 \text{ nHz} = 6.283 \times 10^{-3} \mu\text{rad s}^{-1} = 0.031 \text{ deg day}^{-1}.$$

The typical rotation rate at the solar equator is about 460 nHz ( $2.89 \mu\text{rad s}^{-1}$  or  $14.26 \text{ deg day}^{-1}$ ), and corresponds to a linear velocity  $2012 \text{ m s}^{-1}$ .

## Coefficients of the Solar Rotation Rate in nHz.

$\Omega(\theta) = A + B \cos^2 \theta + C \cos^4 \theta$	Sunspots	Dopp. Features	Magn. Features	Dopp. Shift
A	$469.39 \pm 0.19$	$473.02 \pm 1.59$	$458.22 \pm 0.32$	$453.76 \pm 0.95$
B	$-92.58 \pm 1.94$	$-77.03 \pm 6.05$	$-53.96 \pm 2.07$	$-54.59 \pm 0.80$
C		$-57.46 \pm 8.12$	$-77.19 \pm 3.34$	$-75.44 \pm 1.11$
$\Omega(\theta) = \bar{A} + \bar{B}T_2^1(\theta) + \bar{C}T_4^1(\theta)$				
$\bar{A}$	$450.87 \pm 0.43$	$452.65 \pm 0.95$	$440.87 \pm 0.32$	$433.71 \pm 0.80$
$\bar{B}$	$-18.52 \pm 0.39$	$-23.08 \pm 0.64$	$-21.17 \pm 0.32$	$-21.17 \pm 0.08$
$\bar{C}$		$-2.71 \pm 0.32$	$-3.66 \pm 0.16$	$-3.50 \pm 0.06$

## Rotation rate of various features of the solar surface is different



The surface features (tracers) related to magnetic field structures rotate faster than the solar plasma, rotation of which is measured from the Doppler shift of spectral lines. The magnetic structures are likely to be anchored in the solar interior. Therefore, the faster rotation of these structures probably can be related to higher angular velocity of the interior.

The averaged period of rotation of sunspots is close to 'Carrington period', 25.38 days (the corresponding synodic period is 27.275 days). The Carrington period corresponds to the differential rotation of recurrent (i.e. old) sunspots at approximately 20° latitude, and is often used as a reference.

The surface rotational velocity at this latitude is approximately  $1.9 \times 10^3 \text{ m s}^{-1}$ . It is found that the rotation rate of individual sunspots may vary depending their size, age and relative position to other sunspots of the same group.

## **Helioseismic measurements of solar rotation**

Helioseismology provides a picture of rotation inside the Sun as a function of both depth and latitude. The helioseismic measurements are based on the frequency shift of resonant solar oscillations (modes) due to rotation.

The modes that propagate in the direction of solar rotation have higher frequencies than the modes with the same resonant properties propagating in the opposite direction.

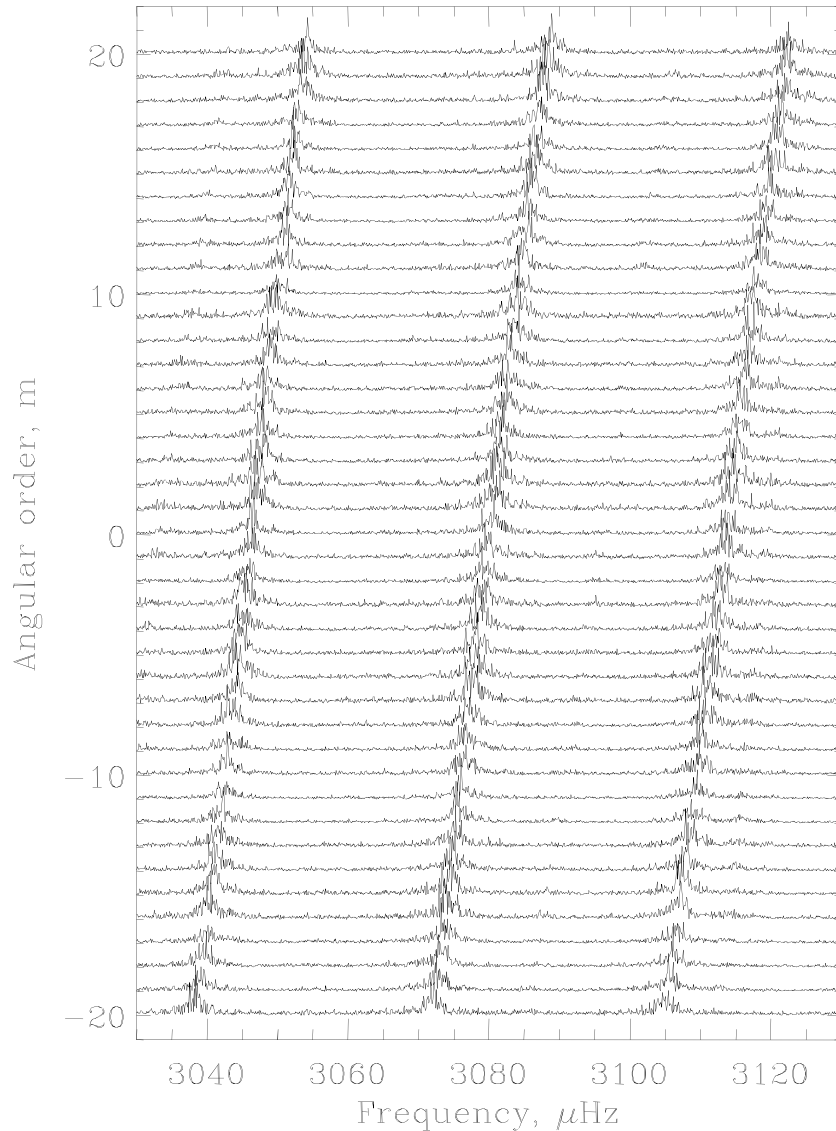
This effect called 'rotational frequency splitting' is somewhat analogous to the Zeeman or Stark splitting of energy levels in atoms. The amount of splitting depends on the rotation rate in the mode resonant cavity and their azimuthal order.

# Rotational frequency splitting

The mode frequencies,  $\nu_{nlm}$ , are characterized by three ‘quantum numbers’: radial order  $n$  which is essentially the number of radial nodes in the mode eigenfunction, and the angular degree  $l$  and azimuthal order  $m$  ( $-m \leq l \leq m$ ) which come from the spherical harmonic part of the eigenfunctions,  $\xi_n(r)Y_l^m(\theta, \phi)$ .

The modes with  $m \neq 0$  represent azimuthally propagating waves. The modes with  $m > 0$  propagate in the direction of solar rotation and, thus, have higher frequencies in the inertial frame than the modes  $m < 0$  which propagate in opposite direction. As a result the modes with fixed  $n$  and  $l$  are split in frequency:  $\Delta\nu_{nlm} = \nu_{nlm} - \nu_{nl0}$ . Thus, the internal rotation is inferred from splitting of normal mode frequencies with respect to the azimuthal order,  $m$ .

## Example of the observed frequency splitting in the oscillation power spectrum



An example of the frequency splitting is shown in this Figure, in which the horizontal curves are oscillation power spectra for  $m = -20, \dots, 20$  of modes with  $n = 15$  and  $l = 19, 20$  and 21. The slope formed by the modal peaks,  $\Delta\nu_{nlm}/\Delta m$ , is approximately the solar synodic rotation rate 0.43  $\mu\text{Hz}$  (or 430 nHz).

# Helioseismic Inverse Problem for Solar Rotation (in-depth topic)

The frequency splitting is proportional to  $m$  and the internal angular velocity  $\Omega(r, \theta)$  averaged over radius  $r$  and colatitude  $\theta$  with splitting kernels  $K_{nlm}(r, \theta)$ :

$$\Delta v_{nlm} = m \int_0^R \int_0^\pi K_{nlm}(r, \theta) \Omega(r, \theta) d\theta dr,$$

where  $R$  is the solar radius. The kernels  $K_{nlm}$  are calculated from mode eigenfunctions.

The determination  $\Omega(r, \theta)$  from a set of observed  $\Delta v_{nlm}$  represent a helioseismic inverse problem. This problem is solved by a regularized least-squares method, or by estimating optimal localized averages  $\bar{\Omega}(r_0, \theta_0)$  from a linear combination of the data:

$$\bar{\Omega}(r_0, \theta_0) = \sum_{nlm} c_{nlm}(r_0, \theta_0) \Delta v_{nlm}$$

where the coefficients  $c_{nlm}$  are chosen to localize the corresponding averaging kernel  $A(r, r_0) \equiv \sum_{nlm} c_{nlm}(r_0, \theta_0) K_{nlm}(r, \theta)$  at the target position  $(r_0, \theta_0)$ .

When the localization attempt is successful the optimally localized kernels resemble a Gaussian, and  $\bar{\Omega}(r_0, \theta_0)$  provides a good estimate of the angular velocity.

## Splitting coefficients (in-depth topic)

The measurements of  $\Delta v_{nlm}$  are often noisy. Therefore, it is useful to determine the frequency splitting in terms of Legendre polynomials in  $m/L$ ,

where  $L = \sqrt{l(l+1)}$  :

$$\Delta v_{nlm} = L \sum_{i=1}^N a_{nl}^{(i)} P_i \left( \frac{m}{L} \right),$$

where  $N$  is typically 5 – 36. (In some recent measurements the Legendre polynomials are replaced with orthogonal polynomials in  $m/L$ .) The odd terms in this equation represent the rotational splitting; the even terms are due to asphericity of the solar structure.

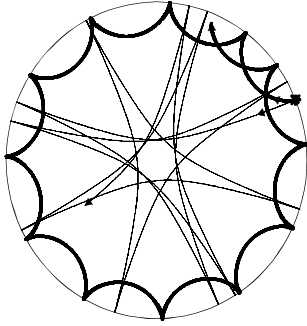
If the internal angular velocity is represented in terms of the orthogonal polynomials with the coefficients depending on radius  $r$  :

$$\Omega(r, \theta) = \bar{A}(r) + \bar{B}(r)T_2^1(\theta) + \bar{C}(r)T_4^1(\theta),$$

then each of the coefficients in this form is primarily determined from the individual  $a_{nl}^{(i)}$  coefficients:

$$a_{nl}^{(1)} = \int_0^R K_{nl}^{(1)}(r) \bar{A}(r) dr, \quad a_{nl}^{(3)} = -\int_0^R K_{nl}^{(3)}(r) \bar{B}(r) dr, \quad a_{nl}^{(5)} = \int_0^R K_{nl}^{(5)}(r) \bar{C}(r) dr.$$

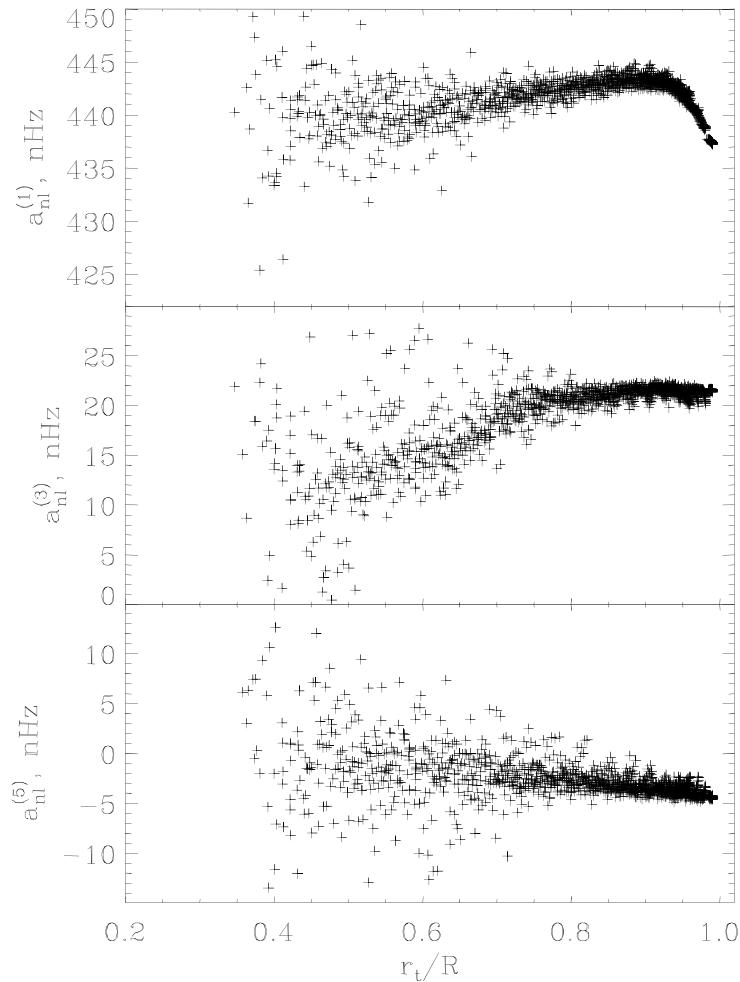
The radial functions  $\bar{A}(r)$ ,  $\bar{B}(r)$ , and  $\bar{C}(r)$  can be determined by solving the one-dimensional integral equations.



For acoustic (p) modes the seismic kernels  $K_{nl}^{(i)}(r)$  are significant mainly inside the mode propagation region,  $(r_t, R)$ , where  $r_t$  is the radius of the inner turning point of acoustic modes,  $R$  is the solar radius.

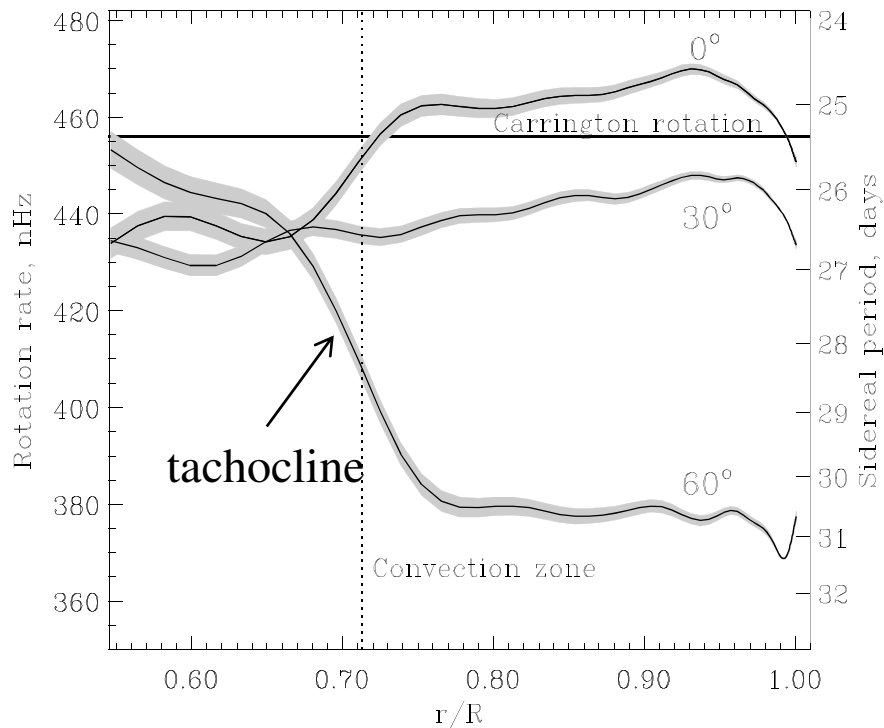
Therefore, it is useful to plot the  $a$ -coefficients as a function of  $r_t$ . Each point in this plot represent an average of  $\bar{A}(r)$ ,  $\bar{B}(r)$ , and  $\bar{C}(r)$  in the interval  $(r_t, R)$ . The first coefficient,  $a^{(1)}$ , which corresponds to the spherically symmetric component of the angular velocity,  $\bar{A}(r)$  shows steep decrease with  $r_t$  near the surface. This corresponds to the increase of the angular velocity with depth in the subsurface layers.

Coefficient  $a^{(3)}$ , which determines the prime component of the latitudinal differential rotation, shows a decrease below  $0.7 R$ , the lower boundary of the convection zone. Therefore, one may expect a decrease of the differential rotation in the radiative zone. This way, investigating seismic data versus the radius of the mode turning points helps to make qualitative inferences without solving the inverse problem.



**Rotational splitting coefficients obtained from a 144-day observing run from SOHO/MDI. The coefficients are plotted as a function of  $r_t$ , the radius of the mode turning point, which is determined for each mode from the relation:  $r_t/c(r_t) = L/2\pi v_{n10}$ , where  $c(r)$  is the sound speed.**

# Results of helioseismic measurements



Rotation rate (in nHz) as a function of radius at the equator ( $0^\circ$ ),  $30^\circ$  and  $60^\circ$  latitude.

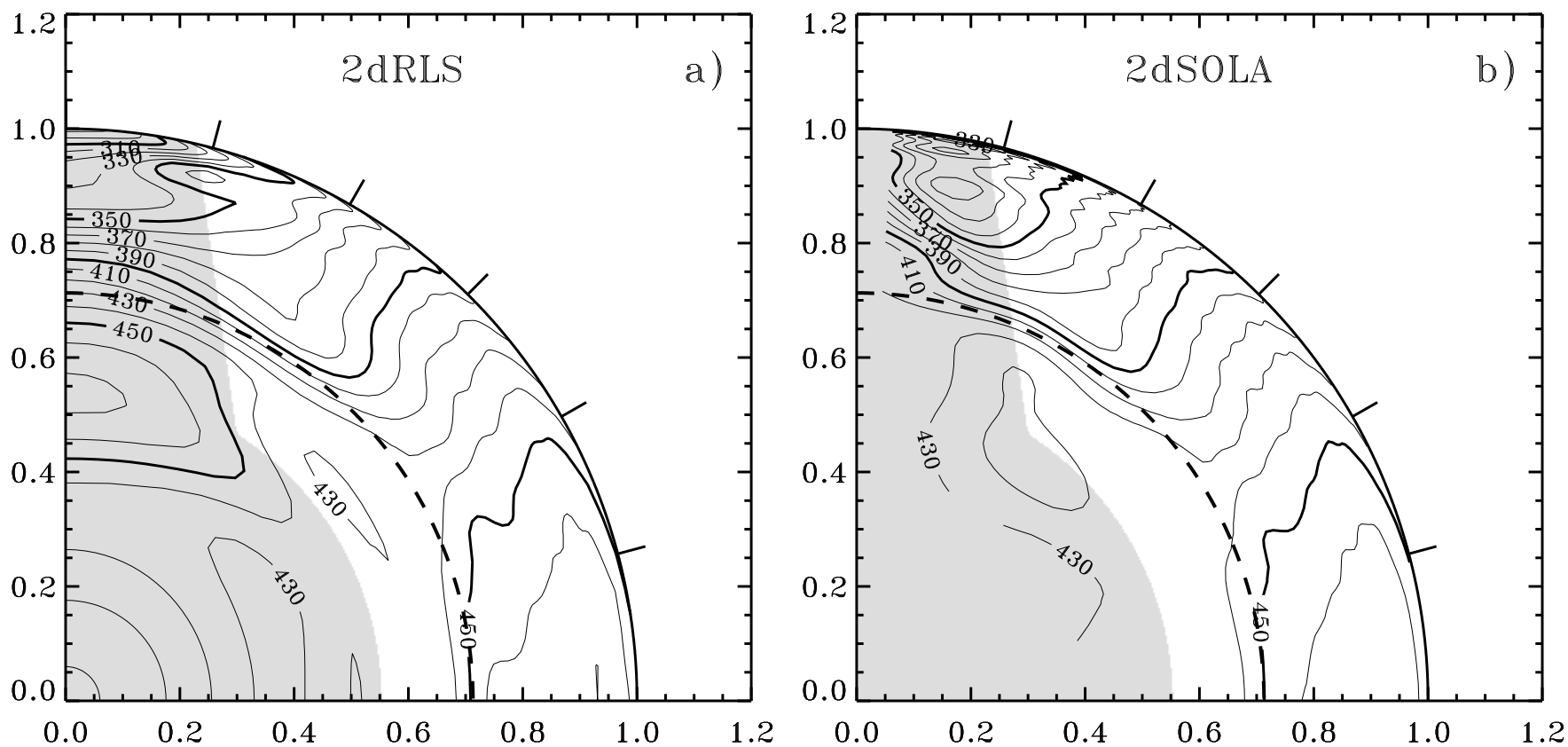
The rotation rate inferred as a function of radius at the equator and at  $30^\circ$  and  $60^\circ$  latitude. It shows that the latitudinal differential rotation extends through the convection zone, and it sharply reduced in the radiative zone.

The layer of transition from the differential rotation in the convection zone to almost uniform rotation in the radiative zone is characterized by a strong radial gradient of the angular velocity at low and high latitudes. This layer is called **tachocline**.

This is probably the place where the global magnetic field of the Sun is generated in the course of the 11-year cycle. Because the internal rotation only in low latitude zones matches the Carrington rotation, it is likely the magnetic field which constitutes sunspots is generated at low latitudes.

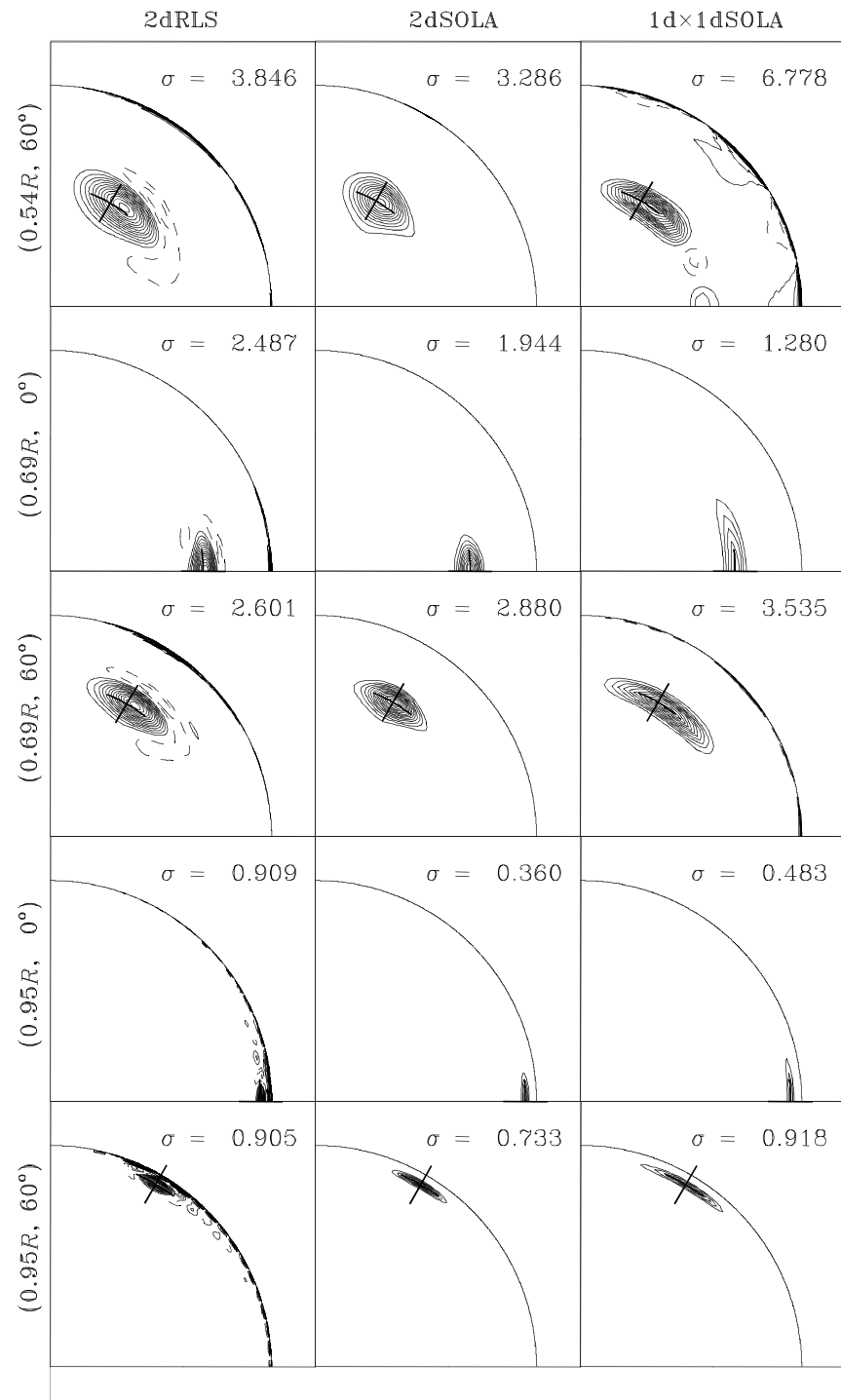
Most of the transition layer is located beneath the lower boundary of adiabatic convection (shown in Fig. by the vertical dashed line) which is located at 0.713 solar radii. The estimates of the width of the transition layer range from 0.05 to 0.1 solar radii.

# Results of helioseismic measurements by two different inversion methods

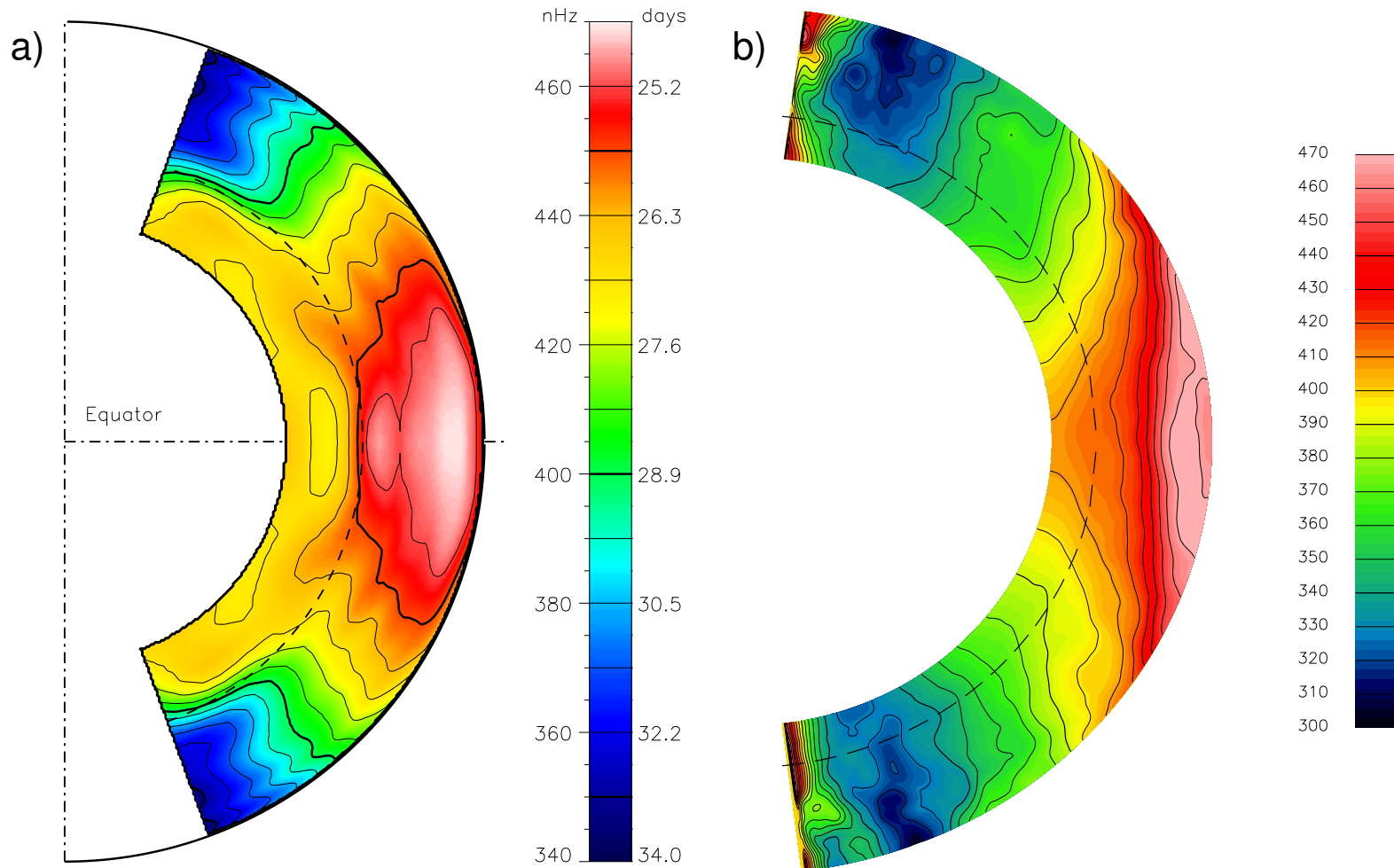


Rotation in the central and near polar regions (shaded areas) has not been determined reliably. However, the current helioseismic data are consistent with the assumption that the central core rotates with the same rate as the outer part of the radiative zone, which is approximately 440 nHz (corresponding period 26.3 days).

# Localized averaging kernels for rotation inversion



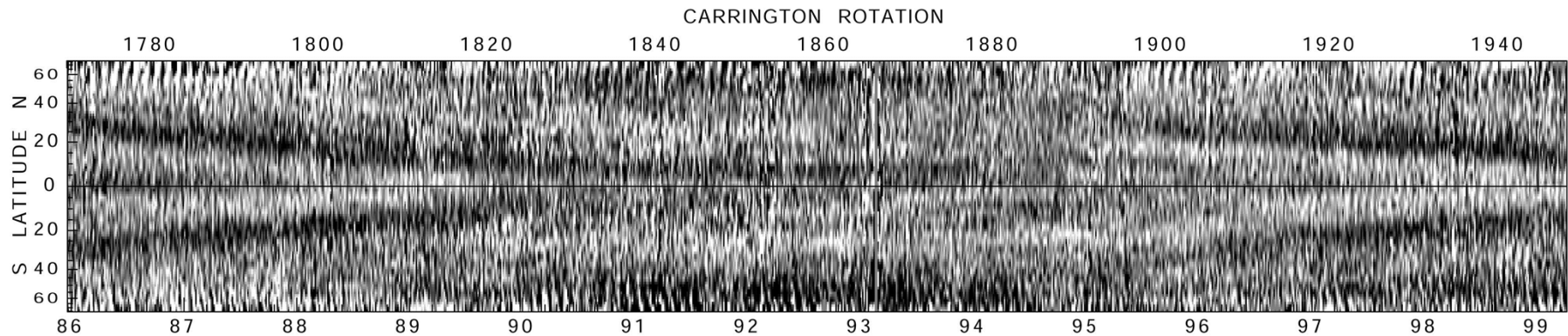
# Comparison of the rotation rate from helioseismology (left) and numerical simulations (right)



- a) Angular velocity profile  $\Omega(r, \theta)$  as deduced from helioseismology, using the GONG data.  
b) Time-averaged angular velocity profile obtained by a 3D simulation. The dashed lines indicate the lower boundary of the convection zone. (Elliott et al. 1998)

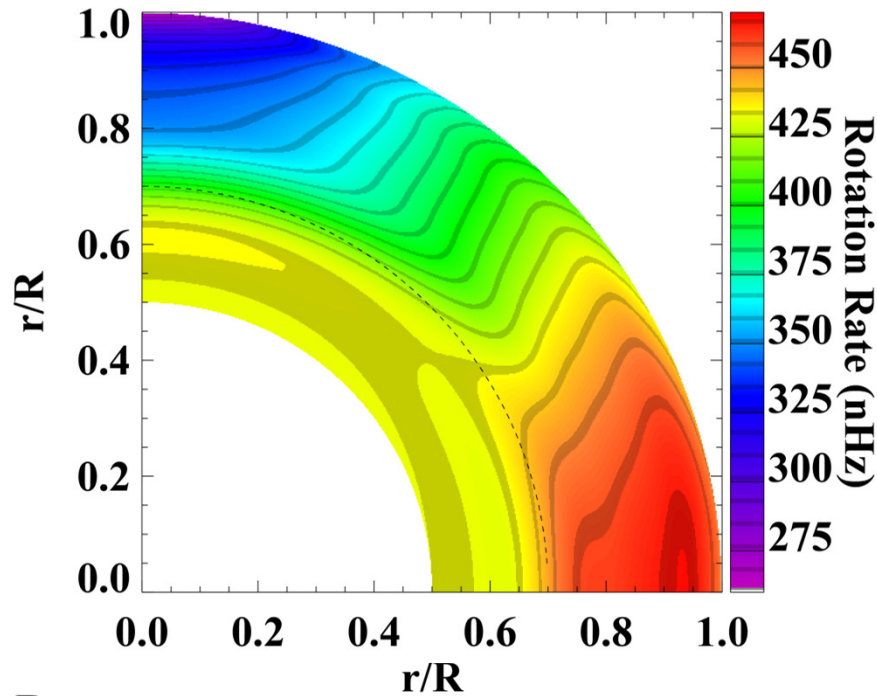
## Migrating Zonal Flows – “Torsional Oscillations”

Observations of solar rotation have also revealed temporal variations related to the 11-year activity cycle. These include an increase in the equatorial rotation rate at solar minimum, and *a migrating pattern of zonal flow bands, known as ‘torsional oscillations’*. The bands start at high latitudes near solar maximum and approach the solar equator at solar minimum.



Migrating zonal flows (‘torsional oscillations’) in 1986-99, determined from the Mt. Wilson Observatory data. The vertical scale is latitude in degrees, the lower horizontal axis shows time in years starting from 1986, and the upper scale shows the corresponding Carrington rotation numbers. The dark areas correspond to the rotational velocity 7.5 m/s faster than the average velocity, and the light area shows rotation 7.5 m/s slower than the average. (Ulrich 1998)

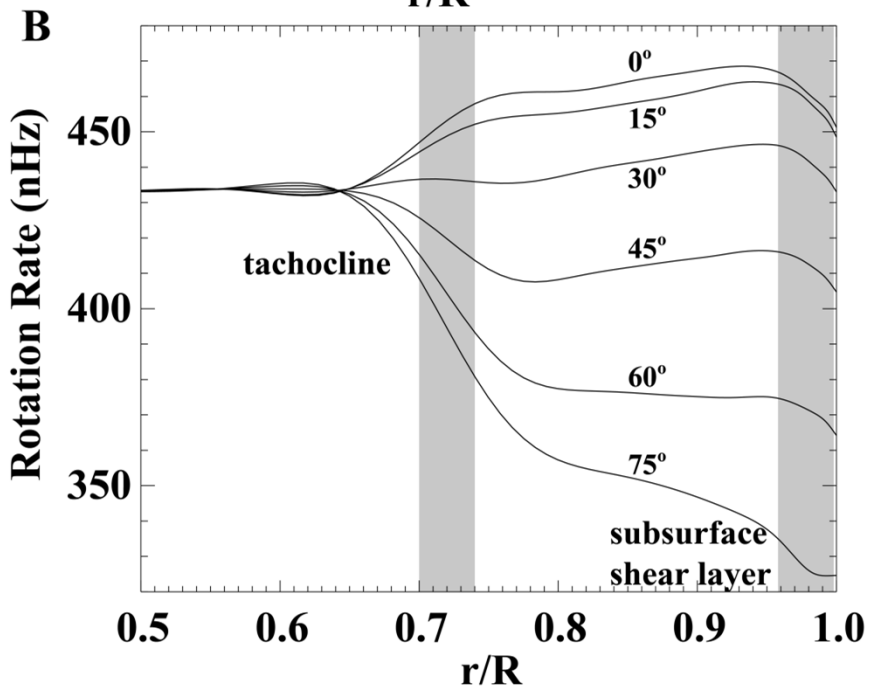
### A Mean Rotation: 2010 – 2018

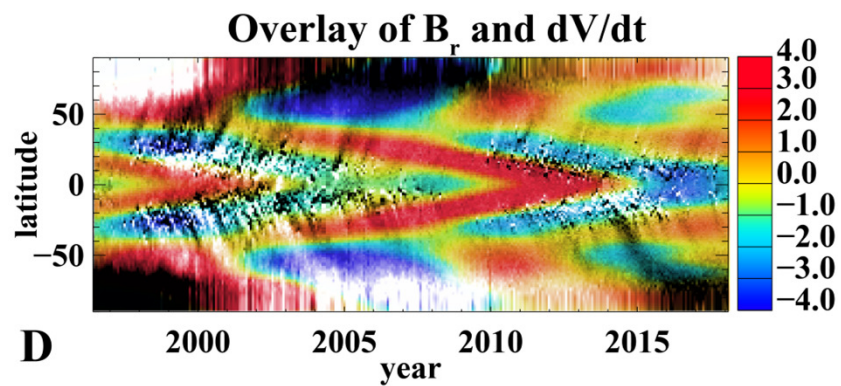
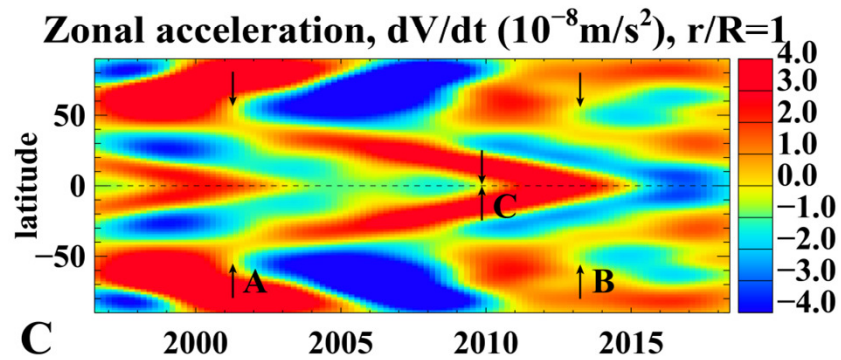
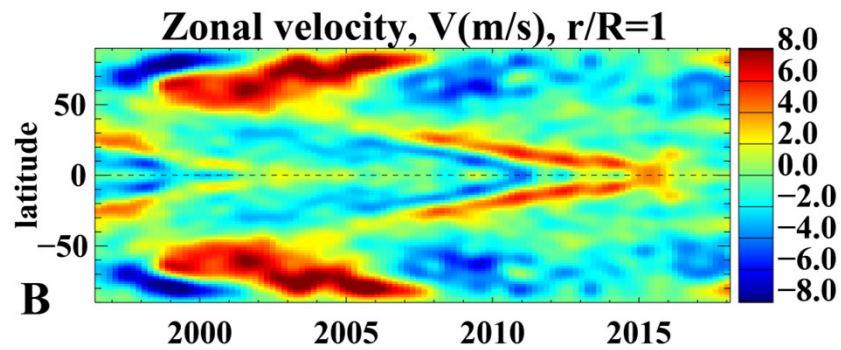
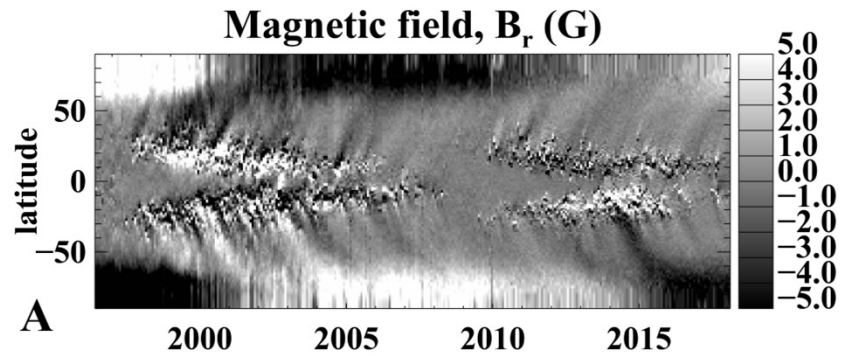


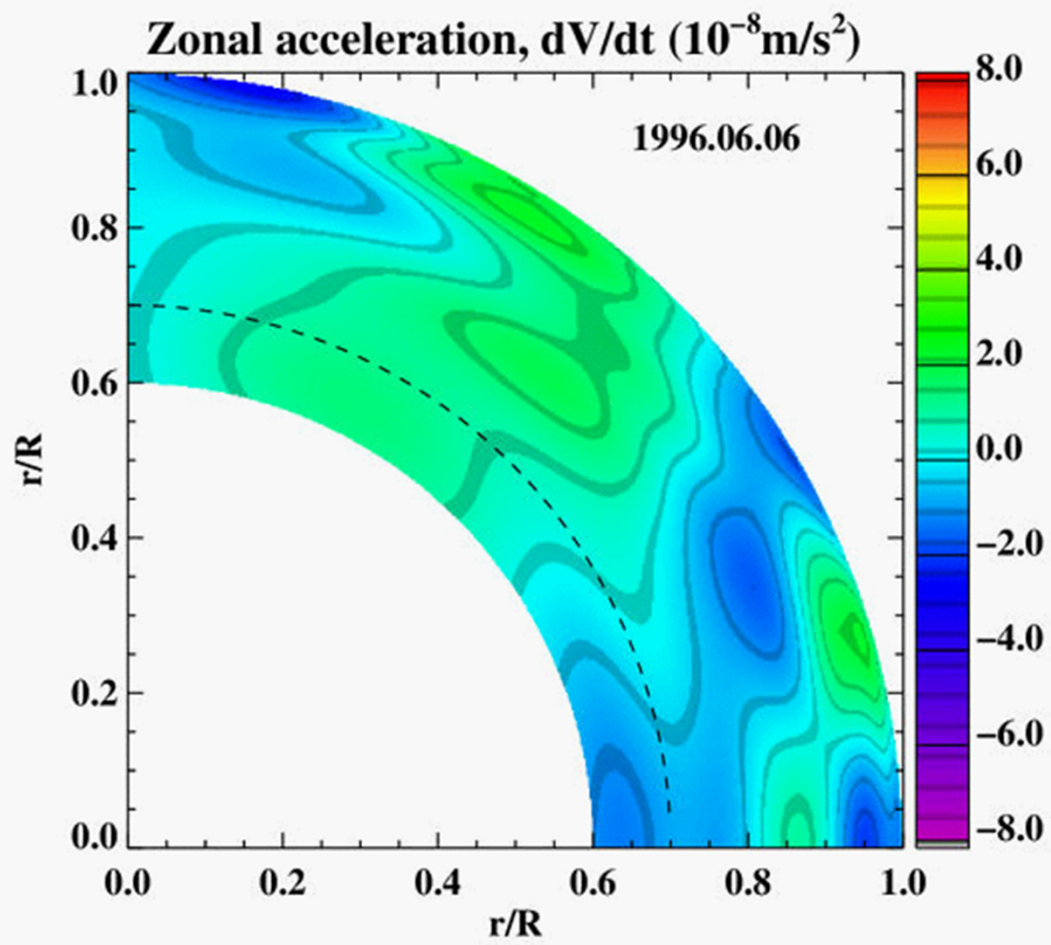
### Dynamo Wave Patterns Inside the Sun Revealed by Torsional Oscillations

Alexander G. Kosovichev,  
Valery V. Pipin, 2018

<https://arxiv.org/abs/1809.10776>

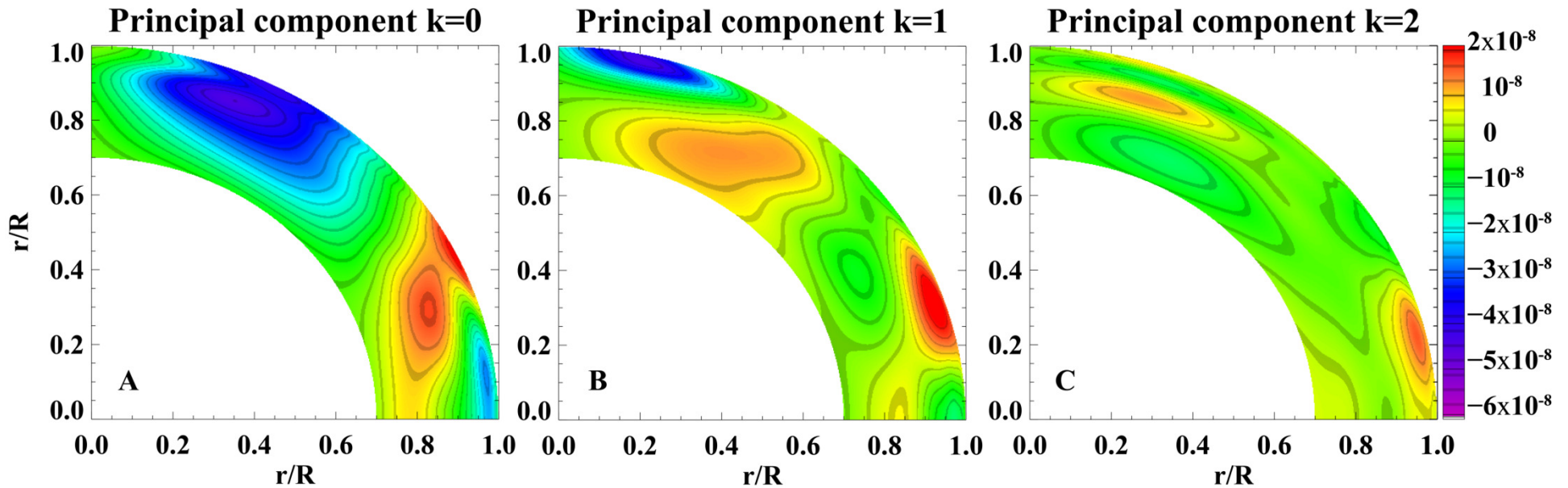
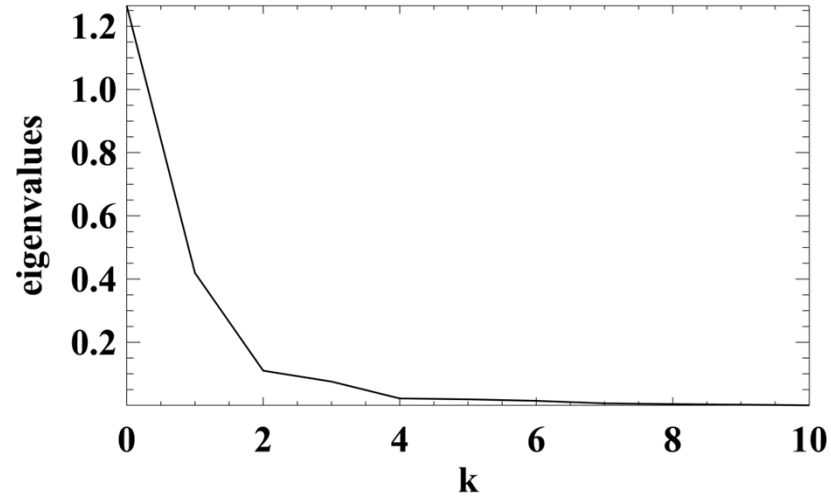




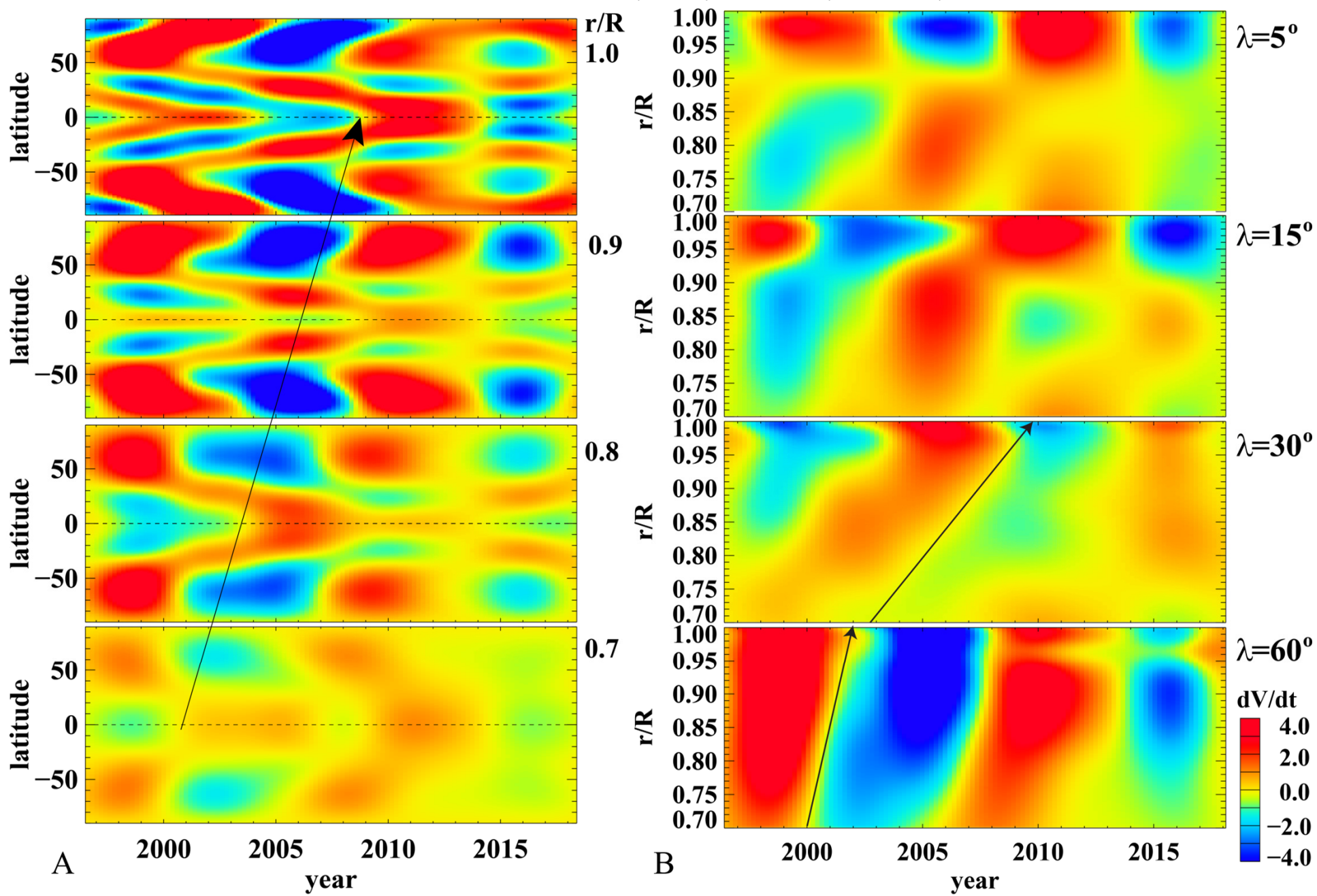


# Principal component analysis

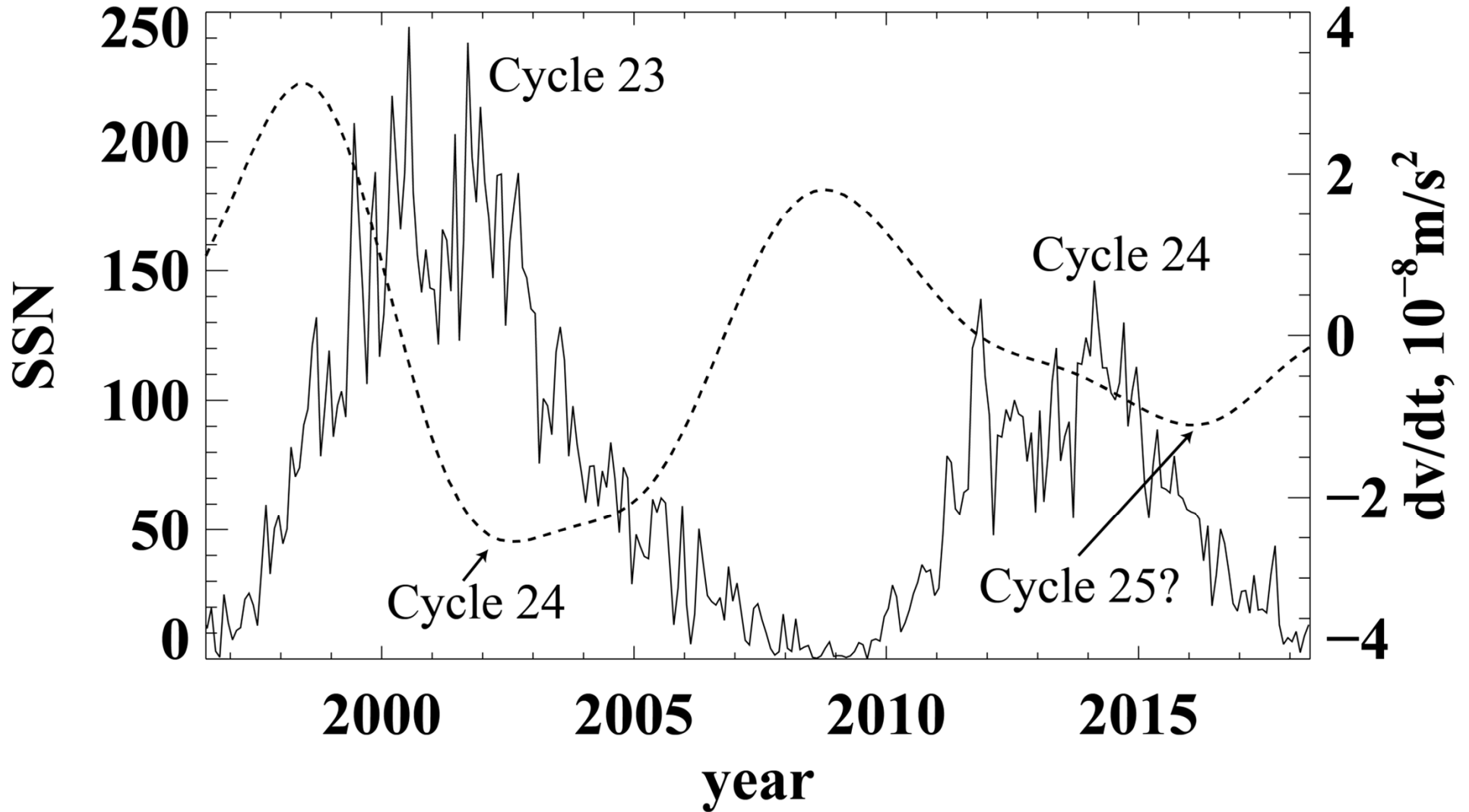
PCA eigenvalues



# Zonal acceleration (PCA), $dV/dt$ ( $10^{-8}m/s^2$ )



# Sunspot number and $dv/dt$ ( $r=0.76R, \lambda=60^\circ$ )

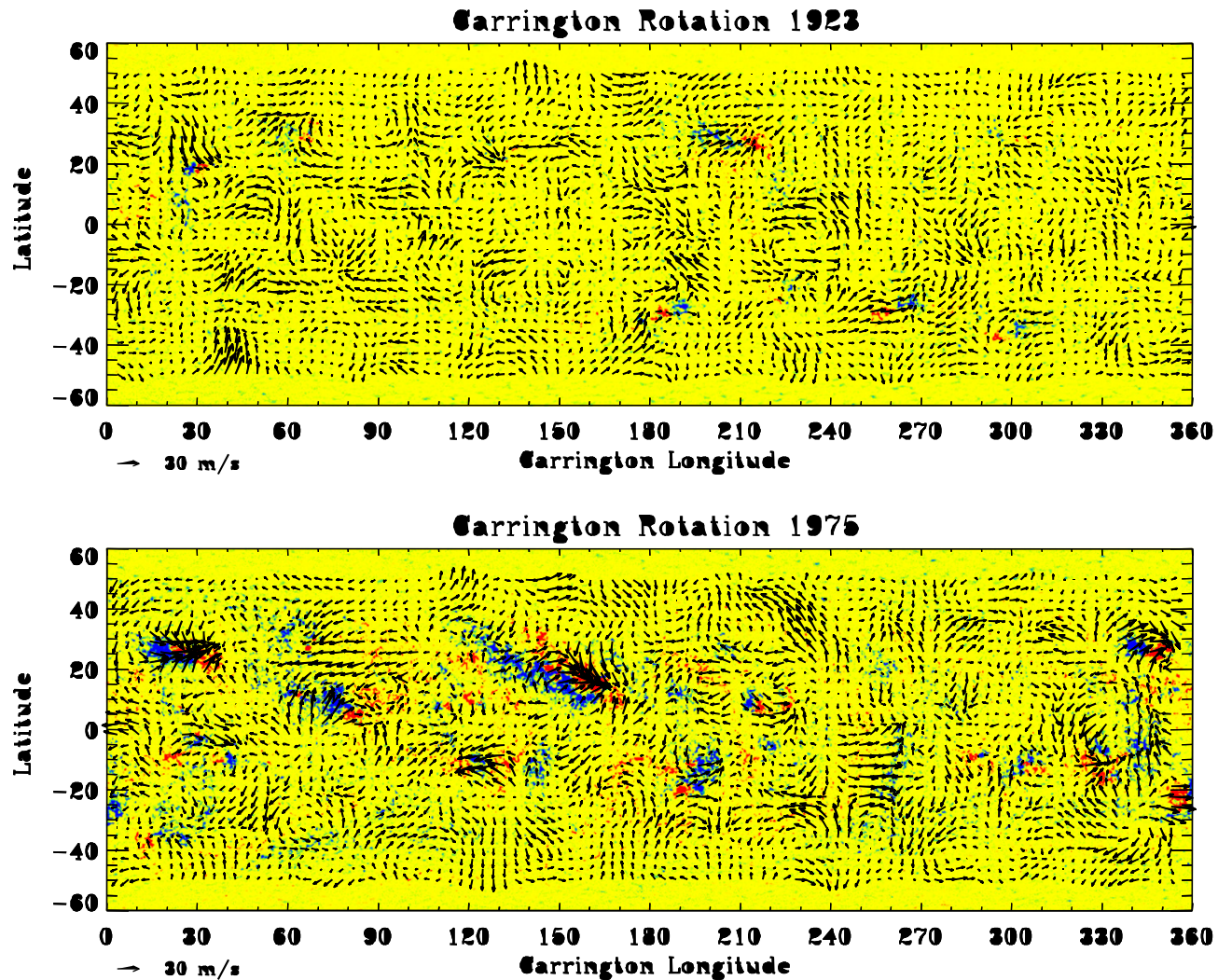


## **North-South asymmetry**

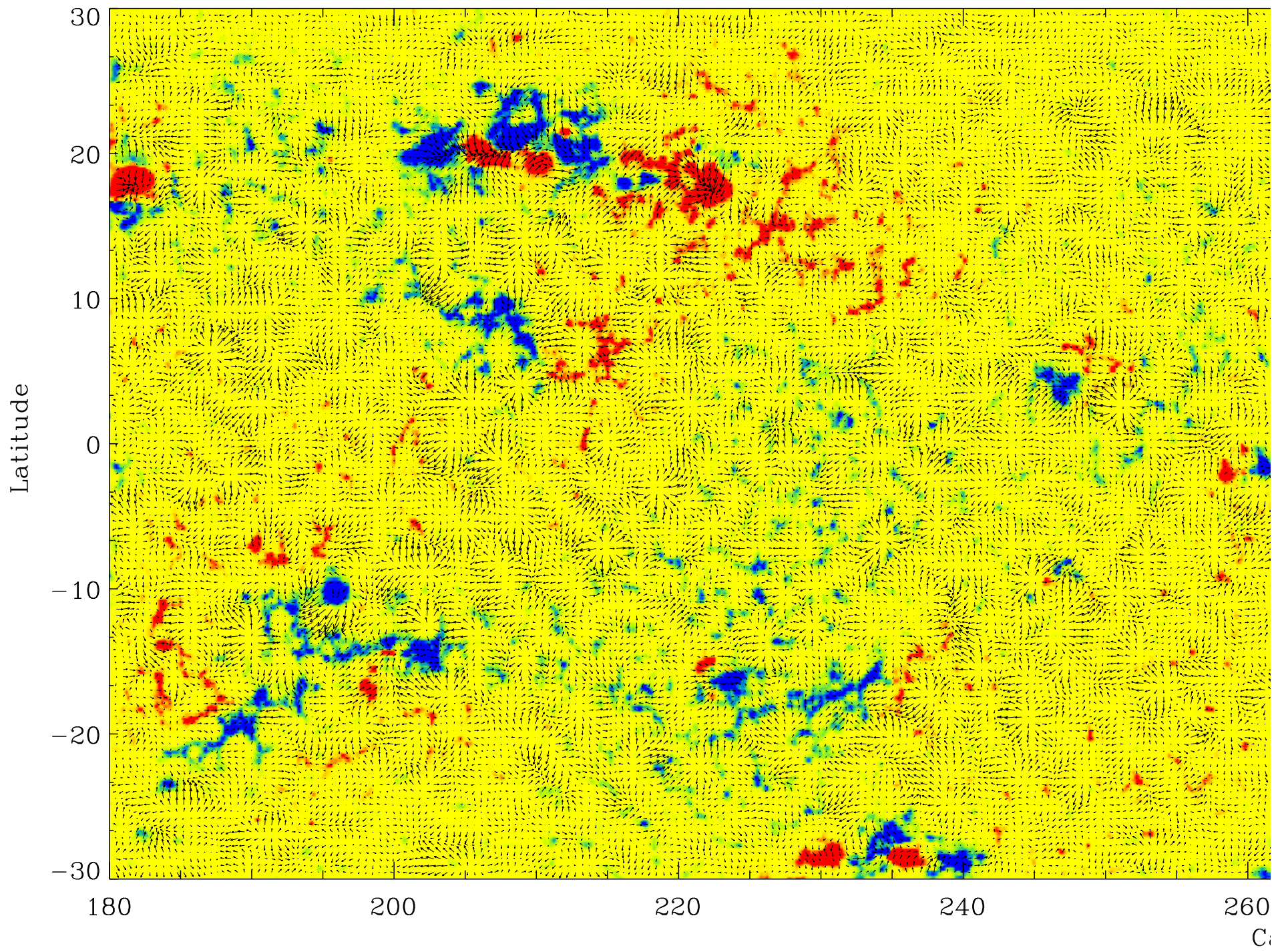
The observation of rotation of sunspots reveal the asymmetry in the differential rotation between the Northern and Southern hemispheres. The asymmetry correlates with the magnetic activity, - the hemisphere with lower activity (fewer active regions) rotates faster than that with higher activity. The difference in the average rotation rate may reach 0.5‰ (10 m/s).

The rotational frequency splitting is not sensitive to the asymmetrical component of rotation to the first order. However, the North-South asymmetry of the internal rotation can be studied by methods of the local helioseismology, e.g. ring-diagram and time-distance techniques.

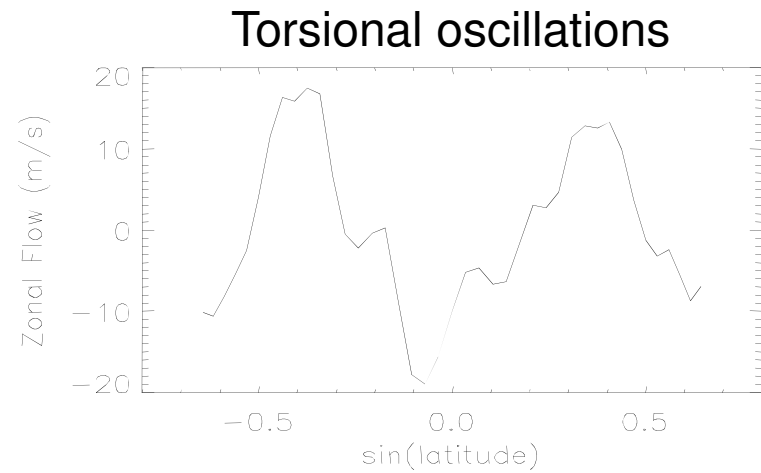
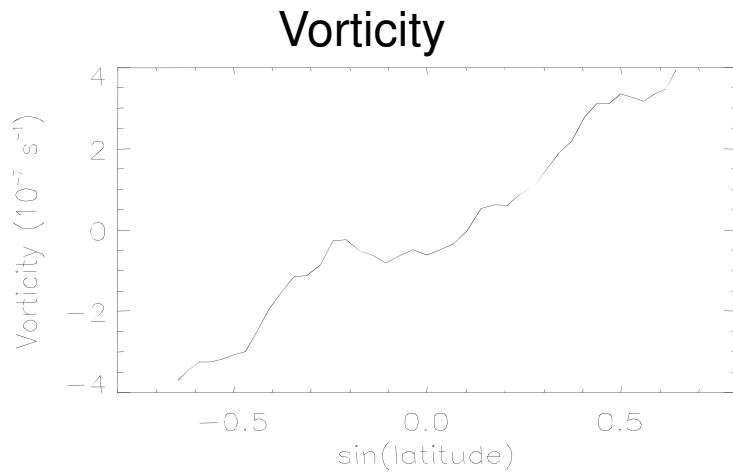
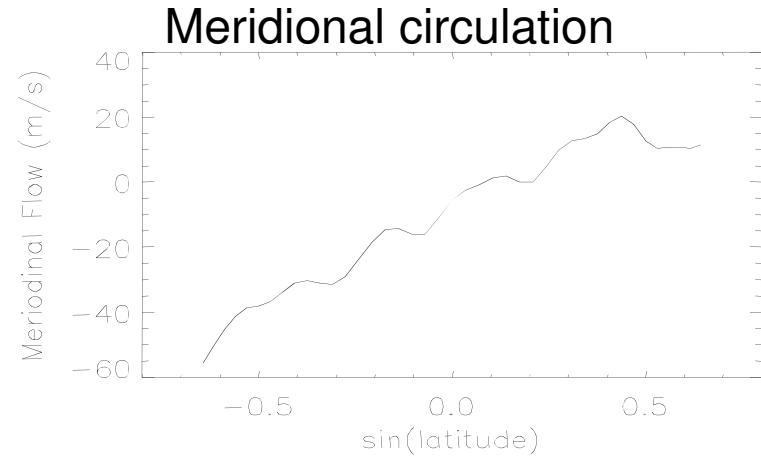
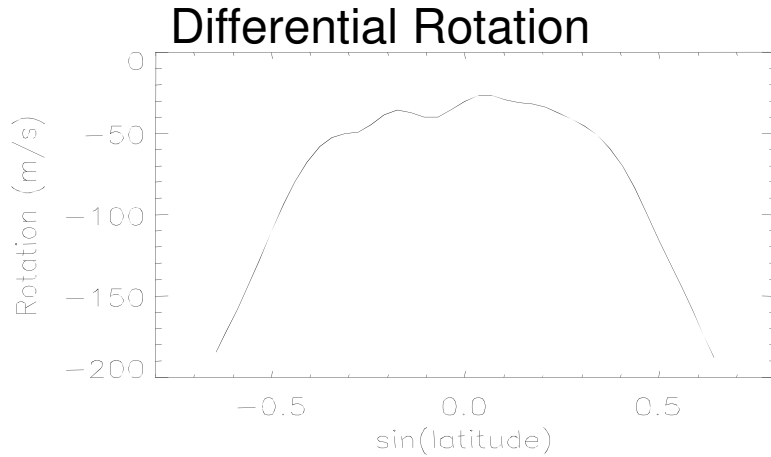
# Measurements of solar rotation by averaging the N-S velocity component over longitude



Zhao, J. & Kosovichev, A.G., 2003



# Differential rotation, meridional circulation, vorticity, torsional oscillations



# Evolution of solar rotation

The Sun rotates more slowly, by a factor of 50, than early main sequence stars. This suggests that the Sun initially rotated much faster and lost most of its angular momentum over its lifetime. The angular momentum of the Sun is believed to be lost due to ‘magnetic braking’ through the magnetized solar wind.

The magnetic braking is mainly applied to the surface layers of the Sun. However, it also slows down the interior because the deep layers of the Sun are coupled with the surface because of convective flows and internal magnetic field. In fact, the convection zone reacts to the external torque almost like a solid body because the angular momentum transport provided by turbulent friction is very efficient.

In the radiative zone the angular momentum transport is much less efficient. It primarily depends on the diffusion transport coefficient. This coefficient may be increased due to various instabilities due to strong shearing motions, excitation of internal gravity waves in the convective overshoot layer or due to instability arising from the accumulation of the  $^3\text{He}$  at the outer boundary of the energy-generating core. The internal magnetic field provides probably the most efficient mechanism of angular momentum transport. Even a very weak magnetic field ( $\approx 10^{-6}$  Gauss) is sufficient to slow down the solar core.

The theories of the evolution of solar rotation based on these mechanisms predicted that the central core might still rotate much faster than the surface. However, the recent helioseismic data have provided strong evidence that the whole radiative zone rotates almost uniformly. Which of the mechanisms of angular momentum transport provide most of the braking is not yet understood.

# Mechanisms of the differential rotation

The angular momentum transport which causes the differential rotation of the convection zone results from thermal stresses and the anisotropic Reynolds stresses,  $Q_{ij} = \langle u_i u_j \rangle$ , where  $u_i$  and  $u_j$  are components of the convective velocity, and the brackets mean an average over longitude  $\phi$ . The anisotropy arises due to the distortion of convection motions by the Coriolis force and other effects of rotation such as the meridional circulation.

The evolution of the angular momentum in the convection zone is described by non-linear Navier-Stokes equations. The coupling between the turbulent solar convection and rotation can be modeled in the 'mean-field' approximation by averaging over an ensemble of convective elements, or by solving the equations numerically on a high-resolution spatial grid. Both approaches are still far from the real situation.

The mean-field theory employs various approximations and assumptions for the mean anisotropic properties of convection, whereas the numerical models do not provide sufficient spatial resolution to model the fully developed turbulent convection accurately.

## Theory of the differential rotation (in-depth topic)

The basic mechanisms of the differential rotation can be understood from the conservation of the angular momentum in the convection zone. Consider the velocity field,  $v$ , in the convection zone as a superposition of the mean rotational velocity,  $\langle v_\phi \rangle$ , a meridional circulation,  $v_m = (\langle v_r \rangle, \langle v_\theta \rangle)$ , and the convective velocity,  $u$ :  $v = \langle v \rangle + u$

From the azimuthal component of the equation of conservation of momentum:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho} \nabla P - \nabla \Phi,$$

where  $\rho$  is the density,  $P$  is the pressure, and  $\Phi$  is the gravitational potential; and the conservation

of mass:  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,$

we obtain the equation of the conservation of the angular momentum,  $\rho s^2 \Omega$ .

$$v = \langle v \rangle + u; \quad \langle v \rangle = v_\phi = r \sin \theta \Omega = s \Omega; \quad s \rho \frac{\partial v_\phi}{\partial t} + s v_\phi \rho \nabla v = 0;$$

$s v_\phi \nabla \rho v = 0$ ; - neglect density fluctuations:  $\frac{\partial \rho}{\partial \phi} = 0$ ;

$$\frac{\partial (\rho s v_\phi)}{\partial t} + \nabla \cdot (s \rho v_\phi v) = 0.$$

Finally,  $\frac{\partial}{\partial t} (\rho s^2 \Omega) + \nabla \cdot (\rho s^2 \Omega v_m + \rho s \langle u_\phi u \rangle) = 0,$

where  $s = r \sin \theta$ , and  $\Omega = \langle v_\phi \rangle / s$ . (Fluctuations of density which may be also important are not taken into account)

$$\frac{\partial}{\partial t}(\rho s^2 \Omega) + \nabla \cdot (\rho s^2 \Omega v_m + \rho s \langle u_\phi u \rangle) = 0,$$

where  $s = r \sin \theta$ , and  $\Omega = \langle v_\phi \rangle / s$ .

This equation shows that the density of angular momentum,  $\rho s^2 \Omega$ , changes either because angular momentum is transported by the meridional circulation,  $v_m$ , or because of the Reynolds stresses,  $\langle u_\phi u \rangle$ . These two processes play important roles in the mechanism of the differential rotation, and, therefore, it is very important to study them. The meridional circulation and convective flows are studied by measuring the Doppler velocity of the solar plasma on the surface and inside the convection zone by time-distance helioseismology.

The numerical simulations have been able to reproduce the differential rotation qualitatively. However, there are significant differences. The simulations provide an angular velocity profile with the contour lines parallel to the rotation axis in the plot, meaning the angular velocity is constant on cylindrical shells, whereas the observations reveal a more complicated profile. The ‘cylindrical’ distribution of the angular momentum is attributed to large-scale convective cells oriented North-South, so-called ‘banana cells’. This type of convection results from the effect of the Coriolis force. Evidently, some other important physical processes are missing in the simulations. Perhaps, the small-scale turbulent convection which is not resolved in the current simulations transports angular momentum quite differently yielding other classes of mean flows and rotation profile.

The future efforts in our understanding of solar rotation will be focused on the precise determination of the rotation rate of the solar core, tachocline, near-polar regions, and the upper convective boundary layer. It is very important to study variations of the solar rotation, both temporal and spatial, which are associated with the solar cycle, and, in particular, the variations in the tachocline and zonal flows (‘torsional oscillations’) because these studies will help to understand the mechanism of solar activity.

## Effects of Stellar Rotation

### The von Zeipel Paradox (in-depth topic)

One consequence of solar rotation is that the equation of hydrostatic equilibrium has to be modified to take centrifugal forces into account. In this case we have

$$\nabla P = -\rho g_{\text{eff}},$$

where  $g_{\text{eff}}$  includes both gravity and centrifugal terms.

The centrifugal acceleration for angular velocity  $\Omega$  at a distance  $s$  from the rotation axis is  $\Omega^2 s \vec{e}_z$ , where  $\vec{e}_z$  is a unit vector perpendicular to the rotation axis.

If the angular velocity depends only on  $s$ , that is constant on cylinders, then the centrifugal acceleration can be expressed in terms of a potential  $V$  :

$$\Omega^2 s \vec{e}_z = -\nabla V$$

We can then combine the gravitational ( $\Phi$ ) and centrifugal potentials:

$\Psi = \Phi + V$ , where  $V = -\int_0^s \Omega^2 s ds$ , and rewrite the equation of hydrostatic equilibrium as

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$$\nabla P = -\rho \nabla \Psi.$$

This means that the pressure gradient and potential gradient are parallel. Therefore, isosurfaces  $\Psi = \text{const}$  correspond to surfaces of constant pressure, therefore pressure is a function of  $\Psi$  only. If the star is chemically homogeneous  $\mu = \text{const}$  then temperature  $T$  is also only a function of  $\Psi$ , if not then  $T/\mu$  is constant on equipotential surfaces. This follows from the equation of state,  $P = R\rho T/\mu$ .

If we take the curl of the equation of hydrostatic equilibrium

$$\nabla P = -\rho \nabla \Psi.$$

then we find that

$$0 = -\nabla \Psi \times \nabla \rho,$$

showing that density is also constant on equipotentials.

For the Poisson equation we have

$$\nabla^2 \Phi = 4\pi G \rho,$$

or

$$\nabla^2 \Psi = \nabla^2 \Phi + \nabla^2 V = 4\pi G \rho - 2\Omega^2.$$

Now consider the equation of radiative energy transport:

$$\vec{F}_R = -\frac{16\sigma T^3}{3\kappa\rho} \nabla T = -\frac{16\sigma T^3}{3\kappa\rho} \frac{dT}{d\Psi} \nabla \Psi,$$

or

$$\vec{F}_R = f(\Psi) \nabla \Psi, \quad \text{where} \quad f(\Psi) = -\frac{16\sigma T^3}{3\kappa\rho} \frac{dT}{d\Psi}.$$

Then, the energy balance equation  $\nabla \cdot \vec{F}_R = \rho \mathcal{E}$ ,

where  $\mathcal{E}$  is the energy generation rate, take form

$$\begin{aligned}\nabla \cdot \vec{F}_R &= \frac{df}{d\Psi} (\nabla\Psi)^2 + f(\Psi) \nabla^2\Psi = \\ &= \frac{df}{d\Psi} (\nabla\Psi)^2 + f(\Psi)(4\pi G\rho - 2\Omega^2) = \rho\mathcal{E},\end{aligned}$$

or

$$\frac{df}{d\Psi} \vec{g}_{\text{eff}}^2 + f(\Psi)(4\pi G\rho - 2\Omega^2) = \rho\mathcal{E},$$

where  $\vec{g}_{\text{eff}} = \nabla\Psi$  is an effective gravity acceleration modified by centrifugal acceleration. In this equation all terms except  $\vec{g}_{\text{eff}}$  are function of  $\Psi$  only. But,  $\vec{g}_{\text{eff}}$  is not constant along the equipotentials because the effective gravity is greater at the poles than at the equator.

**Therefore, the energy balance equation is not satisfied for a uniformly rotating star. A uniformly rotating star cannot be in the thermal equilibrium. This is the von Zeipel paradox.**

**Therefore, the energy balance equation is not satisfied for a uniformly rotating star. This is the von Zeipel paradox.**

**The way out comes from the realization that in a rotating star there is another mechanism of energy transport. This is meridional circulation.**

The characteristic time of meridional circulation is:

$$\tau_{\text{circ}} \approx \frac{GM^2}{LR} \frac{1}{\zeta} \approx \frac{\tau_{KH}}{\zeta},$$

where  $\zeta = \frac{\Omega^2}{2\pi G \rho_c}$  - parameter describing the importance of rotation,  $\tau_{KH}$  is the Kelvin-Helmholtz time. For the Sun,  $\tau_{KH} \approx 10^7$  years,  $\zeta = 10^{-5}$ , therefore  $\tau_{\text{circ}} \approx 10^{12}$  years. The global meridional circulation is not important. However, it is important in near-surface layers where density is small.